Optimal timing of relocation

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Abstract

Purpose – This paper aims to focus on the problem of the optimal relocation policy for a firm that faces two types of uncertainty: one about the moments in which new (and more efficient) sites will become available; and the other regarding the degree of efficiency improvement inherent to each one of these new, yet to be known, potential location places.

Design/methodology/approach – The paper considers the relocation issue as an optimal stopping decision problem. It uses Poisson jump processes to model the increase in the efficiency process, where these jumps occur according to a homogeneous Poisson process, but the magnitude of these jumps can have special distributions. In particular it assumes that the magnitudes can be gamma-distributed or truncated-exponential distributed.

Findings – Particular characteristics concerning the expected optimal timing for relocation, the corresponding volatility and the value of the firm under the optimal relocation policy are derived. These results lead also to the conjecture that the optimal relocation policy is robust in terms of distributions of the degree of improvement of efficiency that are considered, as long as the expected values are the same.

Originality/value – The paper provides an innovative approach to relocation problems, using stochastic tools. Moreover, the use of the truncated exponential and the gamma distribution functions to model the Poisson jumps is particularly suitable, given the situation under study. To the authors’ knowledge, this is the first time that this type of setting is used to tackle a real options problem.

Keywords Globalization, Plant location and layout, Decision making, Capital budgeting

Paper type Research paper

1. Introduction

According to the World Commission on the Social Dimension of Globalization, between 1970 and 1991 the net inflows of foreign direct investment, in percentage of gross domestic product (GDP), approximately doubled worldwide. In contrast, between 1991 and 2000, the speed of this process increased dramatically and the corresponding growth was nearly six fold.

Nowadays, globalization tends to be understood as one of the most important questions faced by developed economies. Business attitudes are increasingly dictated
by international competitiveness, and perceived differences in terms of location attractiveness have lead to levels of relocation previously unknown.

In effect, one of the main consequences of globalization has been a significant wave of production relocation from technology-rich countries towards low labor cost countries. Historically, the costs associated to business communications were relatively high and the corresponding quality was fairly poor, when compared with today. Consequently, the ability to become fully aware of the competitors’ advantages generated by a better choice in terms of location, or by a best set of production practices, and to quickly implement or even to surpass those actions was, on average, not so developed as it is today. The competitive pressure towards the increase in efficiency was naturally less significant and the relatively higher margins earned in a good number of industries provided cushions that could be used to postpone difficult decisions, like those that involve the relocation of a company activities, usually implying job cuts. In addition, lack of information tended to increase perceived uncertainty and this fact was likely to generate further precautions in relation to relocation movements.

Recent increases in the quality of business communications and information in general have been accompanied by a significant decrease in the corresponding costs. Cheap and readily available information imposes the need to make swift decisions and simultaneously reduce uncertainty, creating an environment prone to relocation.

However, in spite of this recent trend, historically, the volume of relocations has apparently followed a slow and gradual course. Regardless of the noticeable appeal, in terms of costs of production, presented by locations in South America and Asia, when compared to the corresponding alternatives in Western Europe and in the USA, for years the pace of relocation was relatively sluggish. Even today, albeit the above mentioned speed increase, the general notion is that a significant number of production sites kept in Europe or in the USA offer disadvantages, when compared to existing alternatives in Eastern Europe or Asia. In fact, the decision process that leads to relocation seems to present significant delays. But, the questions that remain to be answered are: what leads to these delays? What is the dynamic underlying the process of relocation?

Given the natural rational behavior associated with business decision makers, the answers need to be related to the uncertainties facing companies that take these types of decisions. The relocation of a factory or of a big service sector office tends to be carefully analyzed. First of all, because it normally implies high levels of new investment. Additionally, it also tends to create noteworthy costs to layoff current workers and to generate major tribulations (and costs) in terms of public image. However, once the investment is made, if the decision proves to be unwise and the project fails, almost all these costs will be sunk. Consequently, this type of decision tends to have irreversible characteristics and should be analyzed with prudence. The possibility of new developments capable of making a new location economically unattractive will naturally tend to be seriously considered. The relative appeal of each location is a function of the market environment, the comparative ability and cost of the workforce to handle the production technology in question, the level of political incentives to investment and also of the logistic constraints inherent to the products and markets involved. Or, to put it differently, the decision to relocate is influenced by perceived uncertainties in terms of the potential speed with which new more appealing
locations will be made available and by the rate of increase in the corresponding economic efficiency. Both phenomena have stochastic characteristics, and should be modeled properly using an economic model of relocation.

The academic interest in the location of economic facilities has existed for a long time. However, the focus on relocation has not been explored much, as Sleuwaegen and Pennings (2002) point out. A significant segment of the literature related to location comes from the field of international economics, and has focused mainly on the comparative advantages that might lead companies to prefer a certain location to others available. Special emphasis has been placed in the analysis of effects originated by different labor costs and political barriers, as in Motta and Thisse (1994), Cordella and Grilo (1998) and Collie and Vandenbussche (1999). The potential impact of international ventures is also studied in Buckley and Casson (1998), Reuer and Leiblein (2000) and Miller and Reuer (1998a, b), but again the focus is not on the rationale for the decision to move from one location to the other, but on the corresponding consequences in terms of risk and return. Conversely, a different approach to the topic has been used in finance. Capital budgeting studies have considered mostly frameworks in which production might shift from place to place inside a business organization, as a function of the evolution of the real exchange rates between different economies (see, e.g. De-Meza and van der Ploeg, 1987; Capel, 1992; Kogut and Kulatilaka, 1994a, b; Botteron et al., 2003). Underlying these studies is the notion that flexibility regarding the place where to locate production creates value. This flexibility is conceptually regarded as an option, and valued using simple real options frameworks.

However, to our knowledge, in contrast to this paper, none of these studies focuses specifically on the perceived increase in efficiency needed to justify the decision to relocate from one place to another. The rationale underlying the relocation decision is modeled here using a real options framework, in which a firm adopts an optimal behavior in order to identify a single change in location. In accordance to McDonald and Siegel (1986) and Dixit and Pindyck (1994), the decision maker will need to consider not only the advantages associated to the increased efficiency provided by newly available locations, but also the costs associated to the loss of the option to relocate. In effect, all potential locations for a business venture are already known. However, given political, institutional, geographic (accessibility) and economic constraints, frequently it is not economically rationale for some industries to establish facilities in certain locations. Consequently, we will use the concept of access to a location to mean the economic viability of its rational use. Locations will be identified and distinguished in accordance to their efficiency. Increased levels of efficiency will materialize in the ability to generate more output with the same level of input (i.e. due to increased technical competence by the workforce) or to use lower levels of input to generate the same output (i.e. due to lower wages). In every case, the overall result will present smaller average cost of production. The investment cost will be assumed constant and, as mentioned above, a single change in location will be considered. The modeling approach is akin to that used by Farzin et al. (1998), Huisman (2000) and Huisman and Kort (2003) to tackle to adoption of new technologies. The arrival of new information regarding the availability of new locations with improved levels of efficiency is modeled as a Poisson arrival process. The use of the truncated exponential and the gamma distribution functions to model the Poisson jumps is particularly suitable, given the situation under study. To our knowledge, this is the first time that this type of setting has been used to tackle a real options problem.
Along the paper we assume a risk-neutral firm, with a constant discount factor, \( r \), as in Dixit and Pindyck (1994), and we let \( \pi(.) \) denote the cash flow of the firm. We analyze a dynamic model with an infinite planning horizon, and we assume that when the firm chooses a new location, it incurs a sunk cost investment \( I \).

The remainder of the paper is organized as follows. Section 2 presents the model of a firm tackling a relocation decision and facing a stochastic environment, where information about new potentially superior locations is modeled according to a stochastic process. Using the classical framework of Dixit and Pindyck (1994), we present, in Section 3, the decision problem associated with the relocalization decision. Taking into consideration uncertainties related to the speed at which new, more efficient locations, will become known and the rate of increase in the corresponding efficiency, in Section 4 we derive the value of the firm, the optimal switching level and some characteristics of the time until relocalization for two particular cases of the probability law that might rule the jumps in the efficiency process, namely the truncated-exponential and the gamma distributions. Section 5 provides the corresponding numerical results and the parallel economic rationale. Concluding remarks are provided in Section 6.

A word about notation used in this paper and notably concerning random variables. If \( X \) is a random variable, we denote its distribution function by \( F_X(.) \) and its density function by \( f_X(.) \). Moreover, \( \text{IE}[X] = \int udF_X(u) \) denotes its expected value, whereas \( \text{IE}[X^2] = \int u^2dF_X(u) \) is its second order moment and \( V[X] \) its variance, with \( V[X] = \text{IE}[X^2] - \text{IE}^2[X] \). The indicator function, \( 1_a \) is defined as follows: if proposition \( a \) is true, then the value is one; otherwise it is zero. We use the symbol \( \Box \) to denote end of a proof of a lemma or a theorem. Finally w.p.1. means with probability one and i.i.d. means independent and identically distributed.

2. Stochastic framework

In this section we develop the model of a risk-neutral firm tackling a relocation decision and facing a stochastic environment, where information about new, potentially superior, locations is modeled according to a stochastic process. The approach to capital budgeting decision making is assumed optimal.

Nowadays, all potential locations are geographically known. However, this does not mean that every location is available in economic terms. Some potential locations are not effectively accessible due to political and environmental reasons, others because the technology available cannot be rationally used by the workforce on hand, or because they are difficult to access. The economic environment changes continuously at these and other levels that are relevant to the relative economic appeal of the different location sites. With the passing of time, new sites become known and accessible. In order to be competitive they need to be increasingly appealing and gradually more efficient.

In economic terms, neither the increase in efficiency associated to each new location nor the time that will elapse between the moments in which two sequent efficiency optimizing locations become available are known in advance. Thus we are facing a situation with two levels of uncertainty, the first level corresponding to the time at which a new location becomes available, and the second level to the subsequent increase in the efficiency.

Assume that \( \theta(t) \) denotes the efficiency associated with the best location available at time \( t \), with \( \theta(0) \) denoting the initial efficiency (i.e. the efficiency associated with the initial location). Moreover \( \Theta = \{\theta(s), s \geq 0\} \) denotes the corresponding stochastic
process. Now independently of the firm, new (and more efficient) locations can become available. Suppose that at time $t$ a new location becomes available. Then the process $\Theta$ has a jump at that time $t$.

If $U_i$ denotes the (random) value of the $i$th jump of the process $\Theta$, then $\{U_i, i \in \mathbb{N}\}$ denotes simply the sequence of random jumps of the process $\Theta$. We assume that this is a sequence of positive i.i.d. random variables, identically distributed to the random variable $U$, and we denote its distribution function by $F_U(\cdot)$. The fact that the jumps are positive random variables implies, in particular, that $\theta(t_1) \leq \theta(t_2)$, for $t_1 < t_2$ w.p.1. This last assumption means that we consider only locations that improve, in some degree, the efficiency of the firm.

Given the uncertainty inherent to the diverse factors capable of affecting the potential economic attractiveness of different locations, and the impending unstable relationships between those factors, it is reasonable to admit that we are not dealing with a learning process with embedded continuous leaps in terms of information arrival. Therefore, it makes sense the use of an exponential distribution (characterized by the referred propriety of absence of memory) to model the time that elapse between the moments in which two consecutive efficiency optimizing locations becomes available. We note that these assumptions are usually assumed not only in the finance literature (e.g. Merton, 1976; Beckers, 1981; Ball and Torous, 1983; Carr et al., 2002), but also in some others that involve mathematical applications, as in Benedek and Villars (2000).

We note that if, in addition to the assumption of exponential times, we assume that new locations become available independently of the firm, then the counting process associated with these new locations is a Poisson process (Ross, 1996, 2005). Therefore we model the first level of uncertainty using the following setting: we assume that more efficient locations become available according to a Poisson process $N = \{N(t), t \in \mathbb{R}^+\}$, of rate $\lambda$, where $N(t)$ denotes the number of efficiency improving locations that become available in the time interval $(0, t]$, and we assume that $N$ evolves independently of the firm.

Thus the process $\Theta$ is a jump process, where jumps are driven by a Poisson process. This means, in particular, that the efficiency process can take a jump upward at random points in time, with positive probability. Consequently, from the previous assumptions, it follows that for any $t \in \mathbb{R}^+$, $\theta(t)$ can be written as follows:

$$\theta(t) = \theta(0) + \sum_{i=1}^{N(t)} U_i$$  

and therefore $\Theta$ is a double Poisson process (Ross, 1996).

We note that this problem is in close connection with the so-called secretary problem (Freeman, 1983), although there are some remarkable differences. For example, in the secretary problem there is a fixed number of items to be presented, and if the last item is presented it must be accepted. In the relocation problem the number of available locations that come across the decision problem is not a priori known, and thus there is not the obligation regarding any decision about the choice of an available location. In addition, the time at which new locations become available is also random, as we assume that the time that elapses between consecutive events has exponential distribution (and thus the number of efficiency improving locations that become available in the time can be modeled as a Poisson process).
Next we introduce the sequential problem associated with the decision of relocation.

3. The decision problem

In this section we describe in more detail the decision problem. We follow closely notation and results presented in Farzin et al. (1998), Huisman (2000), Huisman and Kort (2003) and Couto (2006), and references therein. For this reason we skip most technical results and proofs.

It is clear from the description of the problem that the decision to relocate can be stated as a capital budgeting decision problem. Each time a new (and more efficient) location becomes available, the firm has to decide if it stays in the same place (avoiding an investment cost, that we denote by $I$, but loosing the opportunity to produce more efficiently) or if it changes to the new location. In fact, the firm faces the dilemma of continuing in the present location or stop, and move to a new location. This decision strongly depends on the relationship between the current efficiency of the firm and the efficiency that it will achieve in the new location and the investment costs. In order to justify a change in location, the corresponding efficiency gains need to overcompensate the resultant relocation costs.

Thus the problem can be restated as follows: there exists a critical value $\theta^*$ such that $\theta(t) > \theta^*$, the firm decides to invest (stopping action) in this new location, whereas if $\theta(t) < \theta^*$, the optimal decision is to continue in its current site, and wait for other locations to become available. Following Dixit and Pindyck (1994), we call the value $\theta^*$ the optimal switching level.

Moreover, let $T^*$ denote the following (random) variable:

$$T^* = \inf\{t > 0 : \theta(t) \geq \theta^*\}.$$  \hspace{1cm} (2)

If the firm always acts optimally, then $T^*$ is simply the time of relocation of the firm.

We note that $T^*$ and $N(T^*)$ are related according to the following equation:

$$P(T^* \leq t) = P(N(t) \geq N(T^*))$$  \hspace{1cm} (3)

where equation (3) holds because $N$ is a Poisson process (Ross, 1996); moreover $T^*$ is a stopping time (Ross, 1996), for the Poisson process $N$.

Finally, let $V(\theta)$ denote the value of the firm after relocation (to a spot with efficiency value $\theta$), and let $F(\theta)$ denote the value of the firm when its current efficiency is $\theta$.

In this paper we assume (without discussion) the necessary conditions on $V$ in order to ensure the existence and uniqueness of $\theta^*$. The optimal switching level $\theta^*$ can be found using the so-called value matching condition. The value matching condition is the result of matching the value of the unknown function of the firm with the embedded options with those of the known termination payoff function. In that precise point the decision maker is indifferent to stay on the current location or decide to invest in the new one.

If the firm decides to change its current location at time $t$, then it means that for all $\theta > \theta^*$:

$$F(\theta) = V(\theta) - I,$$  \hspace{1cm} (4)

which is precisely the termination payoff, as we assume at most one relocation.
On the other hand, if the firm does not change its current location, i.e. if \( \theta \leq \theta^* \), then the value of the firm when the present efficiency is \( \theta, F(\theta) \), is given by:

\[
\frac{\pi(\theta(0))}{r + \lambda} + \frac{\lambda}{r + \lambda} \left[ \int_0^{\theta^* - \theta} F(\theta + u)dF_U(u) + \int_{\theta^* - \theta}^{\infty} (V(\theta + u) - I)dF_U(u) \right]
\]  

where the first component corresponds to the present value of the payoffs inherent to the initial location; the first element in the second component represents the value of the real option to relocate production to a place where a higher level of efficiency will be achieved; and the second element in the second component corresponds to the net present value of the firm after moving to the new location at time \( T^* \) – disbursing the investment cost, \( I \), and benefiting from the increase in net cash flows granted by the increased level of efficiency.

In the next section we derive results concerning the optimal switching level \( \theta^* \), the value of the firm \( F(\cdot) \), and the optimal location time \( T^* \) for two particular classes of random variables, namely the truncated-exponential and the gamma distributions. We note that Huisman (2000), in a different setting, has already presented equivalent results concerning the following density distributions: degenerate, uniform, and exponential. We stress that the results that we present in this paper concerning the truncated-exponential and the gamma distribution are original.

4. Optimal timings of relocation and volatilities

The results presented in the previous section allow for the general valuation of firms facing relocation decisions. However, the corresponding values will depend on the levels of efficiency increase (here denoted by \( U_i \)) and also on dynamics inherent to the process that governs efficiency evolution (namely through the parameter \( \lambda \)). These dynamics will naturally diverge in accordance to the industry and geographical regions considered. In the present article, the above-mentioned dynamics are modeled using different density distribution functions.

Accordingly, in order to study the evolution of the value of the company under an optimal relocation policy for different risk environments and initial locations, in the rest of the paper we consider particular density functions. Namely we consider the following families for \( U \): the truncated-exponential (with parameters \( M \) and \( \mu \)), and the gamma (with parameters 2 and \( \mu \)). For each one we derive the value of the firm, \( F(\cdot) \), the optimal switching level, \( \theta^* \), and the first two order moments for the time of adoption of the new location, \( \mathbb{E}[T^*] \) and \( \mathbb{E}[(T^*)^2] \), respectively. In fact, for managerial decision purposes, the period of time that a firm might expect to stay in the current premises prior to relocation is an especially relevant piece of information, since it affects most of the short term operating decisions that will be taken. Something similar might be stated in relation to the corresponding volatility.

We remark that, as \( T^* \) is a non-negative random variable, then it follows that \( \mathbb{E}[T^*] = \int_0^{\infty} uF_{T^*}(du) = \int_0^{\infty} (1 - P(T^* \leq t))dt \), Ross (1996), where \( F_{T^*} \) denotes the distribution function of \( T^* \). In view of this result and of equations (1) and (2), it follows that:

\[
\mathbb{E}[T^*] = \int_0^{\infty} (1 - P(T^* \leq t))dt = \int_0^{\infty} P(\theta(t) < \theta^*)dt.
\]
and therefore:

$$\text{IE}[T^*] = \int_0^\infty P\left( \theta(0) + \sum_{n=1}^{N(t)} U_n < \theta^* \right) dt$$

(6)

$$= \int_0^\infty \sum_{k=0}^\infty \left( \sum_{n=1}^k U_n < \theta^* - \theta(0) \right) P(N(t) = k) dt$$

$$= \int_0^\infty \sum_{k=0}^\infty \left( \sum_{n=1}^k U_n < \theta^* - \theta(0) \right) \frac{e^{-\lambda t} (\lambda t)^k}{k!} dt$$

$$= \int_0^\infty \left[ e^{-\lambda t} + \sum_{k=0}^\infty \int_0^{\theta^* - \theta(0)} f \sum_{n=1}^k U_n (x) dx \frac{e^{-\lambda t} (\lambda t)^k}{k!} \right] dt$$

$$= \frac{1}{\lambda} + \int_0^\infty \left[ \sum_{k=1}^\infty \left( \int_0^{\theta^* - \theta(0)} f \sum_{n=1}^k U_n (x) dx \right) \frac{e^{-\lambda t} (\lambda t)^k}{k!} \right] dt$$

(7)

where \( f \sum_{n=1}^k U_n (.) \) denotes the density function of the sum of independent and identically distributed random variables \(U_1, U_2, \ldots, U_k\). Using similar arguments, one can prove that:

$$\text{IE}[T^*]^2 = \frac{2}{\lambda^2} + \int_0^\infty \left[ \sum_{k=1}^\infty \left( \int_0^{\theta^* - \theta(0)} f \sum_{n=1}^k U_n (x) dx \right) \frac{e^{-\lambda t} (\lambda t)^k}{k!} \right] dt.$$ 

(8)

### 4.1 Truncated-exponential distribution

Historically, in developed economies, productivity levels have revealed a tendency to grow exponentially. Consequently, economic intuition would lead to the use of the exponential distribution to model increases in efficiency due to changes in location. However, in this paper we decided instead to consider the truncated-exponential distribution. As the density function of the exponential distribution is nonzero for all values in \( \mathbb{R}^+ \), if we modeled the increases in efficiency by an exponential distribution, then we would be implicitly assuming an increase in the efficiency that could be arbitrarily large, with non-zero probability. Given the historical background at this level, this possibility does not seem reasonable. It may be appropriate to consider a bounded increase in the efficiency, and therefore the truncated-exponential may be a wise option, as it is bounded w.p.1 by a finite value (that here we denote by \(M\)).

We say that \(U\) has truncated-exponential distribution with parameters \(M\) and \(\mu\) if its density function is as follows:

$$f_U(u) = \frac{\mu e^{-\mu u}}{1 - e^{-\mu M}}, \ u \in (0, M)$$

(9)

where \(M, \mu \in \mathbb{R}^+\). We note that:
\[ \text{IE}[U] = \frac{1 - \frac{Me^{-\mu M}}{1 - e^{-\mu M}}}{\mu^2} \text{, IE}[U^2] = \frac{2}{\mu^2} - \frac{M(2 + \mu M)e^{-\mu M}}{\mu(1 - e^{-\mu M})}. \]

In addition, if we let \( M \to \infty \), then \( U \) is exponentially distributed, with parameter \( \mu \).

In the light of equation (5), it follows that the value of the firm in the continuation region (i.e. when \( \theta < \theta^* \)) is given by the following expression:

\[
F(u) = \frac{\pi(\theta(0))}{r + \lambda} + \frac{\lambda}{r + \lambda} \int_0^{\min(\theta^* - \theta, M)} F(u + \theta) \frac{\mu e^{-\mu u}}{1 - e^{-\mu M}} \, du + 1_{\theta^* - \theta < M} \frac{\lambda}{r + \lambda} \int_0^{M} (V(u + \theta) - I) \mu \frac{e^{-\mu u}}{1 - e^{-\mu M}} \, du. \tag{10}
\]

Therefore the optimal switching level, \( \theta^* \), is the solution of the following equation:

\[
\frac{\pi(\theta(0))}{r + \lambda} + \frac{\lambda}{r + \lambda} \int_0^{M} (V(\theta^* + u) - I) \mu \frac{e^{-\mu u}}{1 - e^{-\mu M}} \, du = V(\theta^*) - I. \tag{11}
\]

Next we present one lemma that allows for the computation of a general solution of

\[
\frac{\pi(\theta_0)}{r + \lambda} + \frac{\lambda}{r + \lambda} \int_0^{M} (V(\theta^* + u) - I) \mu \frac{e^{-\mu u}}{1 - e^{-\mu M}} \, du = V(\theta^*) - I. \tag{12}
\]

(lemma 4.1). Let \( a_0, a_1, a_2 \) and \( a_3 \) be any non-negative real numbers. Then the solution of the equation:

\[
f(x) = a_0 + a_1 \int_0^{a_2 - x} f(x + y) \frac{a_3 e^{-a_3 y}}{1 - e^{-Ma_3}} \, dy \tag{13}
\]

is given by:

\[
f(x) = e^{a_3 x \left(1 + (-1 + \frac{1}{1 - e^{Ma_3}}) a_1\right)} c(1 - e^{-Ma_3}) a_3 \frac{(-1 + e^{Ma_3})a_0}{(1 + e^{Ma_3}(-1 + a_1))^2} \tag{14}
\]

where \( c \) is a constant determined by initial value.

Moreover, the solution of the equation:

\[
f(x) = a_0 + a_1 \left[ \int_0^{a_2 - x} f(x + y) \frac{a_3 e^{-a_3 y}}{1 - e^{-Ma_3}} \, dy + \int_{a_2 - x}^M g(x + y) \frac{a_3 e^{-a_3 y}}{1 - e^{-Ma_3}} \, dy \right] \tag{15}
\]

is given by:
where \( c_1 \) and \( c_2 \) are constants determined by initial values.

In view of Lemma (4.1), if we make the following attributions:

\[
a_0 = \frac{\pi(\theta(0))}{r + \lambda}; \quad a_1 = \frac{\lambda}{r + \lambda}; \quad a_2 = \mu; \quad g(y) = V(y) - I
\]

then we have the following result concerning the value of the firm.

**Theorem 4.2.** The value of the firm when the present efficiency is \( \theta, F(\theta), \) is given by:

\[
\begin{align*}
& e^{\mu \theta} \left( \frac{1}{r + \lambda} \right) \frac{1}{1 - e^{-\mu \theta}} \left[ c_1 - \int_{c_2}^{\theta} G(y)dy \right] \quad \theta < \theta^* \wedge \theta^* - \theta < M \\
& e^{\mu \theta} \left( \frac{1}{r + \lambda} \right) \frac{1}{1 - e^{-\mu \theta}} \left[ c_2(1 - e^{-\mu \theta}) \right] \frac{(1 - e^{\mu \theta})}{(1 - e^{\mu \theta})(1 - e^{-\mu \theta})} \quad \theta < \theta^* \wedge \theta^* - \theta \geq M
\end{align*}
\]

with:

\[
G(y) = \frac{e^{-y + M + (1 - e^{-M \mu}) \frac{1}{r + \lambda}} \left( (1 + e^{-M \mu}) \frac{\pi(\theta)}{r + \lambda} - V(y + M) \frac{\lambda}{r + \lambda} \right) \mu^2}{(1 - e^{-M \mu})^2}
\]

where the constants \( c_1, c_2 \) and \( c_3 \) are the solution of the following system of equations:

\[
\begin{align*}
& c_1 - \int_{c_2}^{\theta} G(y)dy = 0 \\
& e^{\mu \theta} \left( \frac{1}{r + \lambda} \right) \frac{1}{1 - e^{-\mu \theta}} \left[ c_1 - \int_{c_2}^{\theta} G(y)dy \right] = \frac{\pi(\theta)}{r + \lambda} + \int_0^\theta (V(\theta^* + u) - I) \frac{\mu \lambda - \mu \mu}{1 - e^{-\mu \theta}} du \\
& e^{\mu \theta} \left( \frac{1}{r + \lambda} \right) \frac{1}{1 - e^{-\mu \theta}} \left[ c_1 - \int_{c_2}^{\theta} G(y)dy \right] = \frac{\mu \lambda - \mu \mu}{1 - e^{-\mu \theta}} \int_{c_2}^{\theta} G(y)dy
\end{align*}
\]
We stress that equation (17) and the determination of the constants $c_1, c_2$ and $c_3$ is still algebraically much involved, even for simple production functions.

We note that in order to derive the first two order moments of $T^\star$ we need, according to equations (7) and (8), the distribution of $\sum_i U_i$. As the truncated-exponential is not closed under sums of i.i.d. random variables, and as the distribution of the sum does not have a closed form, we cannot derive closed analytical formulas for $\text{IE}[T^\star]$ and $\text{IE}[(T^\star)^2]$. Latter on, on Section 5, we show results obtained using numerical simulation.

### 4.2 Gamma distribution

A gamma distribution with parameters $2$ and $\mu$ can be seen as the sum of two independent and identically distributed exponential random variables, with parameter $\mu$ (Ross, 2005). We note that the investment decisions are subject to several risk factors eventually independent. Therefore, in modeling this type of problems, makes sense to introduce distribution functions that enable the analytical treatment of more than one state variable, in order to approximate the modeling exercise to corporate reality.

Assume that $U$ has a gamma distribution, with parameters $2$ and $\mu$, so that the density function of $U$ is as follows:

$$f_U(u) = \frac{\mu^2}{\Gamma(2)} u e^{-\mu u}, \quad u \in \mathbb{R}^+$$

where $\mu > 0$.

For such a distribution, the two first order moments are given by:

$$\text{IE}[U] = \frac{2}{\mu}, \quad \text{IE}[U^2] = \frac{6}{\mu^2}.$$ 

In the light of equation (5), it follows that the value of the firm in the continuation region (i.e. when $\theta < \theta^\star$) is given by the following expression:

$$F(\theta) = \frac{\pi(\theta(0))}{r + \lambda} + \frac{\lambda}{r + \lambda} \int_0^{\theta^\star - \theta} F(\theta + u) \mu^2 u e^{-\mu u} du$$

$$+ \frac{\lambda}{r + \lambda} \int_{\theta^\star - \theta}^{\infty} (V(\theta + u) - I) \mu^2 u e^{-\mu u} du.$$ 

Thus the optimal switching level, $\theta^\star$, is the solution of the following equation:

$$\frac{\pi(\theta(0))}{r + \lambda} + \frac{\lambda}{r + \lambda} \int_0^{\infty} (V(\theta^\star + u) - I) \mu^2 u e^{-\mu u} du = V(\theta^\star) - I.$$ 

The next lemma provides a result that allows one to compute the solution for the value of the firm.

**Lemma 4.3.** Let $a_0, a_1, a_2$ and $a_3$ be any non-negative real numbers. Then the solution of the equation:

$$f(x) = a_0 + a_1 \left[ \int_0^{a_2 - x} f(x + y) a_3^2 y e^{-a_3 y} dy + \int_{a_2 - x}^{\infty} g(x + y) a_3^2 y e^{-a_3 y} dy \right]$$

is given by:

$$\text{Optimal timing of relocation}$$
\[ f(x) = \frac{a_0}{1 - a_1a_3} + \frac{c_1}{a_3} \sqrt{a_1} e^{a_3(1 + \sqrt{a_1})x} - \frac{c_2}{a_3} \sqrt{a_1} e^{a_3(1 - \sqrt{a_1})x} \]

where \( c_1 \) and \( c_2 \) are such that:

\[ \frac{a_0}{1 - a_1a_3} + \frac{c_1}{a_3} \sqrt{a_1} - \frac{c_2}{a_3} \sqrt{a_1} = a_0 + a_1 \int_{a_2}^{\infty} g(y)a_2^2 ye^{-a_3 y} dy \]

and

\[ \frac{a_0}{1 - a_1a_3} + \frac{c_1}{a_3} \sqrt{a_1} e^{a_3(1 + \sqrt{a_1})a_2} - \frac{c_2}{a_3} \sqrt{a_1} e^{a_3(1 - \sqrt{a_1})a_2} = a_0 + a_1 \int_{0}^{\infty} g(a_2 + y)a_2^2 ye^{-a_3 y} dy. \]

(See Couto (2006) for the proof of this result.)

In view of Lemma 4.3, if we make the following attributions:

\[ a_0 = \frac{\pi(\theta(0))}{r + \lambda}; a_1 = \frac{\lambda}{r + \lambda}; a_3 = \mu; g(y) = V(y) \]

then we have the following result concerning the value of the firm.

**Theorem 4.4.** The value of the firm is given by:

\[
F(\theta) = \begin{cases} 
\frac{\pi(\theta(0))}{r + \lambda} + \frac{c_1}{\mu} \sqrt{\frac{\lambda}{r + \lambda}} e^{\mu(1 + \sqrt{\frac{\lambda}{r + \lambda}})\theta} - \frac{c_2}{\mu} \sqrt{\frac{\lambda}{r + \lambda}} e^{\mu(1 - \sqrt{\frac{\lambda}{r + \lambda}})\theta} & \text{if } \theta \leq \theta^* \\
V(\theta) - I & \text{if } \theta > \theta^* 
\end{cases}
\]  

where \( c_1 \) and \( c_2 \) are the solutions of the following set of equations:

\[
\frac{\pi(\theta(0))}{r + \lambda - \lambda \mu} + \frac{c_1}{\mu} \sqrt{\frac{\lambda}{r + \lambda}} - \frac{c_2}{\mu} \sqrt{\frac{\lambda}{r + \lambda}} = \frac{\pi(\theta(0))}{r + \lambda} + \frac{\lambda}{r + \lambda} \int_{\theta^*}^{\infty} V(y) \mu^2 ye^{-\mu y} dy - \frac{\pi(\theta(0))}{r + \lambda - \lambda \mu} + \frac{c_1}{\mu} \sqrt{\frac{\lambda}{r + \lambda}} e^{\mu(1 + \sqrt{\frac{\lambda}{r + \lambda}})\theta^*} - \frac{c_2}{\mu} \sqrt{\frac{\lambda}{r + \lambda}} e^{\mu(1 - \sqrt{\frac{\lambda}{r + \lambda}})\theta^*} = \frac{\pi(\theta(0))}{r + \lambda} + a_1 \int_{0}^{\infty} (V(\theta^* + y) - I) \mu^2 ye^{-\mu y} dy.
\]

We note that Theorem 4.4, similarly to Theorem 4.2, is still algebraically and numerically quite involved, and very difficult to interpret.

In the next theorem we provide expressions for the computation of the expected value and the volatility of \( T^* \), the time until relocalization.
**Theorem 4.5.** The first two moments of $T^*$ are given by:

\[ \text{IE}[T^*] = \frac{1}{\lambda} + \frac{\mu}{\lambda} \frac{\theta^* - \theta(0)}{2} - \frac{1}{4\lambda} + \frac{e^{-2\mu(\theta^* - \theta(0))}}{4\lambda} \]  

(22)

\[ \text{IE}[(T^*)^2] = \frac{2}{\lambda^2} + \frac{1}{2\lambda^2} \left( -\frac{5}{4} + e^{-2\mu(\theta^* - \theta(0))} \right) \left( \frac{5}{4} - (\theta^* - \theta(0))\mu \right) + \frac{1}{2}(\theta^* - \theta(0))(6 + (\theta^* - \theta(0))\mu) \]  

(23)

**Proof.** Since we assume that $U_i \sim \text{Gamma}(2, \mu)$, it follows that $\sum_{k=1}^{\alpha} U_k \sim \text{Gamma}(2k, \mu)$, as $\{U_i\}$ is a sequence of independent random variables and the Gamma distribution is closed under sums of i.i.d. random variables (Ross, 1996). Therefore, in view of equation (7):

\[ \text{IE}[T^*] = \int_0^\infty \left[ e^{-\lambda t} + \sum_{k=1}^{\infty} \int_0^\infty f(U_i(x))dx \right] dt \]  

\[ = \int_0^\infty \left[ e^{-\lambda t} + \sum_{k=1}^{\infty} \int_0^\infty \frac{\mu^{2k}x^{2k-1}e^{-\mu x}}{(2k-1)!} dx \right] dt \]  

\[ = \int_0^\infty e^{-\lambda t} dt + \int_0^\infty \frac{\mu^{2k}x^{2k-1}}{(2k-1)!} \int_0^\infty \frac{e^{-\lambda t}}{\lambda} dt dx \]  

\[ = \int_0^\infty e^{-\lambda t} dt + \int_0^\infty \frac{\mu}{\lambda} \sum_{k=1}^{\infty} \frac{\mu^{2k}x^{2k-1}}{(2k-1)!} dx \]  

(24)

\[ = \frac{1}{\lambda} + \int_0^\infty \frac{\mu}{\lambda} \frac{e^{-\mu x} \text{Sinh}(\mu x)}{\lambda} dx \]  

\[ = \frac{1}{\lambda} + \int_0^\infty \frac{\mu}{\lambda} \frac{e^{-\mu x} - e^{-\mu x}}{2} dx \]  

(25)

\[ = \frac{1}{\lambda} + \frac{\mu}{2\lambda} \int_0^\infty (1 - e^{-2\mu x}) dx \]  

\[ = \frac{1}{\lambda} + \frac{\mu}{\lambda} \frac{\theta^* - \theta(0)}{2} - \frac{1}{4\lambda} + \frac{e^{-2\mu(\theta^* - \theta(0))}}{4\lambda} \]

and then equation (22) follows. We remark that we use in equation (24) the fact that:

\[ \int_0^\infty \frac{\lambda^{k+1}t^k}{k!} e^{-\lambda t} dt = \int_0^\infty f_{\text{Gamma}(k, \lambda)}(t) dt = 1 \]

(as it is the integral all over the support of the density function) and in equation (25) the fact that $\sum_{k=1}^{\infty} (\mu^{2k}x^{2k-1})/(2k-1)! = \mu \text{Sinh}(\mu x)$ (by definition of hyperbolic sinus), where
Sinh(s) = \frac{e^s - e^{-s}}{2}.

Similarly, it follows from equation (8) that:

\[
\text{IE}[(T^*)^2] = \int_0^\infty \left[ e^{-\lambda \sqrt{t}} + \sum_{k=1}^{\infty} \int_0^\infty \theta^* - \theta(0) \int_0^x U_n(x) dx \cdot \frac{e^{-\lambda \sqrt{t}}(\lambda \sqrt{t})^k}{k!} \right] dt
\]

\[
= \int_0^\infty e^{-\lambda \sqrt{t}} dt + \int_0^\infty \sum_{k=1}^{\infty} \int_0^\infty \theta^* - \theta(0) \frac{\lambda - \mu x}{\mu} \frac{e^{-\lambda \sqrt{t}}(\lambda \sqrt{t})^k}{k!} dt
\]

\[
= 2 \lambda^2 + \int_0^\infty \theta^* - \theta(0) \sum_{k=1}^{\infty} \frac{\lambda - \mu x}{\mu} \frac{e^{-\lambda \sqrt{t}}(\lambda \sqrt{t})^k}{k!} dt.
\]

If in \( \int_0^\infty \left( e^{-\lambda \sqrt{t} \lambda^{k-1}} \sqrt{t} \right) k! dt \) we make the change of variable \( u = \sqrt{t} \), then we get:

\[
\int_0^\infty \frac{e^{-\lambda \lambda^{k-1} \sqrt{t}^k}}{k!} dt = \int_0^\infty \frac{e^{-\lambda \lambda^{k-1} u^k}}{k!} 2udu = 2\text{IE}[\text{Gamma}(\lambda, k)] = \frac{2k + 1}{\lambda}.
\]

Moreover,

\[
\sum_{k=1}^{\infty} \frac{\mu 2^{k} \lambda^{2k-1} (k + 1)}{(2k - 1)!} = \frac{1}{2} \mu^2 x \text{Cosh}(\mu x) + \frac{3}{2} \mu \text{Sinh}(\mu x)
\]

where \( \text{Cosh}(s) = \frac{e^s + e^{-s}}{2} \). Therefore:

\[
\text{IE}[(T^*)^2] = \frac{2}{\lambda^2} + \int_0^\infty \theta^* - \theta(0) \sum_{k=1}^{\infty} \frac{\mu 2^{k} \lambda^{2k-1} (k + 1)}{(2k - 1)!} \frac{2k + 1}{2} \lambda dx
\]

\[
= \frac{2}{\lambda^2} + \int_0^\infty \theta^* - \theta(0) \frac{1}{2} \mu^2 x \text{Cosh}(\mu x) dt
\]

\[
= \frac{2}{\lambda^2} + \mu \frac{\lambda}{2} \int_0^\infty \theta^* - \theta(0) e^{-\mu x} dx
\]

\[
= \frac{2}{\lambda^2} + \frac{1}{2\lambda^2} \left[ -\frac{5}{4} + e^{-2\mu (\theta^* - \theta(0))} \left( \frac{5}{4} - (\theta^* - \theta(0)) \mu \right) + \frac{1}{2} (\theta^* - \theta(0))(5 + (\theta^* - \theta(0)) \mu) \right]
\]

In the next section we illustrate numerically the results derived in this section for particular instances.
5. Comparative statics illustrations

In this section we illustrate some of the results derived in the previous section, using particular instances of truncated-exponential and gamma distributions.

We choose the parameters of the random variables so that they have the same expected value (but different volatility), i.e. everytime a new location becomes available, the expected increase in efficiency will be the same in both cases. In particular, we consider the instances show in Table I.

Therefore we are comparing two situations where the law of the jumps in the efficiency process is different (one is modeled by a truncated-exponential with bound 10, whereas the other is modeled by a gamma-distribution), but in such a way that on average the jumps in the efficiency are equal. Thus with this simple illustration we can check, at least empirically, if the probability law of the jumps in the efficiency is relevant (and in this case we might find significantly different values of optimal switching levels) or, at the opposite, if the expected value of the increase is the most relevant parameter, regardless the probability law. Following Farzin et al. (1998) and Couto (2006), we consider a Cobb-Douglas production function with output elasticity equal to 0.5, output price 1,000, input price 250, discount rate 0.05, and with a sunk cost of investment in a new location \( I = 10,000 \). Currently, the firm operates in a location with efficiency \( u(0) = 1 \), and the rate at which new locations become available is \( \lambda = 0.5 \). Thus the value of the firm after relocation (see Farzin et al. (1998)) is given by:

\[
V(\theta) = 20000\theta^2. 
\]  

(26)

Table II presents numerical results for these situations, concerning:

- The optimal switching level, \( \theta^* \) – obtained through the solution of equation (11), for the truncated-exponential case, and of equation (19), for the gamma case, with \( V(\theta) \) given by equation (26).
- The expected value and variance of \( T^* \) for the gamma case, according to equations (22) and (23), where \( V[T^*] = \mathbb{E}[\langle T^* \rangle^2] - \mathbb{E}^2[T^*] \). We note that for the truncated-exponential case we cannot derive an explicit expression for these two moments (as previously mentioned, the truncated-gamma is not closed under sums of i.i.d. random variables and neither is known its closed form).
- Mean and standard deviation of \( T^* \) (\( \bar{T}^* \) and \( S_{T^*} \), respectively), computed using a sample of 1,000,000 simulations[1].

<table>
<thead>
<tr>
<th>( \theta^* )</th>
<th>( \text{IE}[T^*] )</th>
<th>( \sqrt{\text{Var}[T^*]} )</th>
<th>( \bar{T}^* )</th>
<th>( S_{T^*} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trunc-exp (5, 10)</td>
<td>2.643</td>
<td>–</td>
<td>–</td>
<td>35.01</td>
</tr>
<tr>
<td>Gamma (2, 20)</td>
<td>2.628</td>
<td>34.06</td>
<td>10.05</td>
<td>34.30</td>
</tr>
</tbody>
</table>

Table I.

<table>
<thead>
<tr>
<th>( \text{IE}[U] )</th>
<th>( \sqrt{\text{Var}[U]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trunc-exp (5, 10)</td>
<td>0.1</td>
</tr>
<tr>
<td>Gamma (2, 20)</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table II.
For these two particular situations, the numerical values of both $\theta^*$ and $T^*$ are similar, and therefore one may ask if the optimal relocation policy is robust in terms of the distribution of the jumps in the efficiency process. Note that, if one proves that the optimal policy (in terms of optimal switching level and time until adoption of a new location) depends only on the distribution of the jumps in the efficiency through an expected value, then we can simply discard the information concerning the distribution of the jumps, keeping only in mind this expected value.

Next, we present two plots, showing the behavior of $\theta^*$ as a function of the expected value of $U$, for different values of arrival rate ($\lambda$) of information concerning new locations (Figures 1 and 2).

The optimal policy seems robust in terms of the particular distribution that we use, as the values of $\theta^*$ are nearly the same for the truncated-exponential and for the gamma distributions that we have considered. Additionally, the optimal switching level seems to be essentially an increasing linear function of the level of the expected increase in efficiency (IE($U$)) and of the rate of arrival of information concerning new locations ($\lambda$). See Figure 3 for a particular illustration of the last statement, for the truncated-exponential distribution with expected value 0.5.

Following the above comparative analysis, we also present numerical results concerning the optimal switching level when we impose changes in the above mentioned parameter values. In particular, we analyze the effect on the optimal switching level, $\theta^*$ of the following parameters: input price $w$, investment cost $I$, and discount rate $r$ (see Figures 4-6).
Higher levels of input related costs lead naturally to increases in the value of the option to delay an investment. Therefore, increases in input prices lead to increases in optimal switching levels (see Figure 4). In effect, the level of efficiency that triggers a change in location needs to increase in order to compensate the gross margin reduction, induced by the increase in input prices.

The relationship between optimal switching levels and investment costs follows a similar pattern (see Figure 5). Increases in investment costs need to be properly compensated by efficiency increases in order to justify changes in location.

Finally we present a plot concerning the behavior of the optimal switching level as a function of the discount rate (see plot (6)). Reduced discount rate levels mean smaller time value of money and consequently, a small potential loss for postponing the decision to relocate. In contrast, very high discount rate levels imply untenable
Figure 4.
Behavior of $\theta^*$ for the truncated-exponential distribution and for the gamma distribution as a function of the input price.

Figure 5.
Behavior of $\theta^*$ for the truncated-exponential distribution and for the gamma distribution as a function of the investment cost.

Figure 6.
Behavior of $\theta^*$ for the truncated-exponential distribution and for the gamma distribution as a function of the discount rate.
6. Concluding remarks

The relevance of the subject is transversal to the whole society. The interested public is a large spectrum of social actors. Entrepreneurs/businessmen are concerned about the impacts on the companies’ results. Employees with access to working places and its implications regarding the choice of their place of residence. Finally, public entities, responsible for the promotion of policies that lead to economic and social welfare and to society’s sustained development, connoisseurs of the problem’s magnitude, as much to employees as for businessmen, tend to, obviously, understand its consequences, so that they are able to take the conducting measures to turn obvious its negative repercussions and enhance the positive ones. In the USA the McKinsey Global Institute foresees a growth of 30-40 percent on relocations, until 2009, and the Forrester Research points out the loss of 3.3 million jobs until 2015, in the results of this kind of processes (Drezner, 2004). According to Deloitte, until 2009, 2 millions jobs will be liquidated on the North-American financial sector in presence of relocation processes (Drezner, 2004).

Despite the social problems associated to the relocation phenomenon of economic activities, the signs of slackening are not visible in its development. The economic advantages, which are associated to it are incredibly significant that it will be extremely difficult to doubt its development. A study of the economist Catherine Mann, from the Institute for International Economics, Mann (2000), has estimated that, between 1995 and 2002, relocation has had an annual positive impact equivalent to 0.3 percent of the USA GDP. Likewise, a report from the McKinsey Global Institute points out that for each dollar invested in relocation it creates a profit of 12-14 percent (Drezner, 2004). In conformity, an analysis made based on the predictions of some of the biggest worldwide technological companies has enabled the Gartner Group to conclude that one in ten jobs associated to technologies will be dislocated from the USA for countries with low-priced manpower, until the end of the year. The US Offshore Outsourcing: Structural Changes, Big Impact presented in New York in 2003, also reveals that 500,000, from the 10.3 million jobs associated to technologies, may change for countries like India or China. On the consultant’s opinion, the phenomenon might even widen to jobs associated to sectors like banking, health care or insurance, also related to technology.

In spite of the relevance of the topic, in contrast to this article, no other study that we are aware of focuses on the perceived increase in efficiency needed to justify the decision to relocate from one place to another. Using a dynamic programming framework, we have analyzed the problem of the optimal timing for the relocation of a firm that faces a constant and irreversible level of investment expenditure to move its production site. The modeling framework assumes that the availability of new and more efficient location sites evolves according to a Poisson process. The dynamics of the corresponding efficiency increases are modeled using continuous distributions. Given the specific characteristics of the relocation decision, and unlikely any other work that we are aware of in this field, we have used different distributions for the jumps magnitude, as the truncated exponential and the gamma distribution functions.
The reason to use the truncated exponential is quite intuitive from the practical problem. Considering the historical reality, it is not expected to happen, in a certain moment in time, a huge leap in efficiency which reflects a colossal change, for instance, in the productivity level afforded by the decision of relocating a production unit. Hence, the importance of introducing a limit for the critical level of efficiency which when attained leads to the investment decision. Therefore, in order to prevent possible inconvenient which occur on the study of this subject, it makes sense to consider the same type of exponential behavior, but more sophisticated as in the case of the truncated exponential distribution, in the sense of eliminating potential problems inherent to that distribution function. Hence, we seek to limit the leap in order to stop it from attaining excessive values for the economic reality.

Regarding the reason to consider the gamma distribution: nowadays the investment decisions are subject to several risk factors possibly independent, so it makes sense to introduce in the modeling of this kind of problems distribution functions which allow treating analytically more than one variable of state, in order to approximate the modeling exercises from corporate reality.

According to our research, the optimal timing of relocation is significantly affected by the uncertainties related to both the expected rhythm that characterizes the arrival of information regarding the availability of new, more efficient location sites, and the degree in efficiency improvement from one to the other. Our simulation results suggests the results of our model are in accordance with economic rationale.

Naturally, a simple framework as the one proposed here has some limits that could and should be overcome in future work.

Note
1. Simulations were obtained using the Mathematica software.

References


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