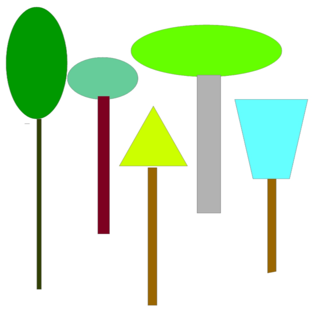
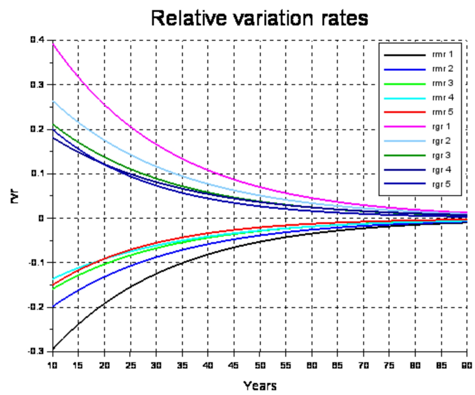
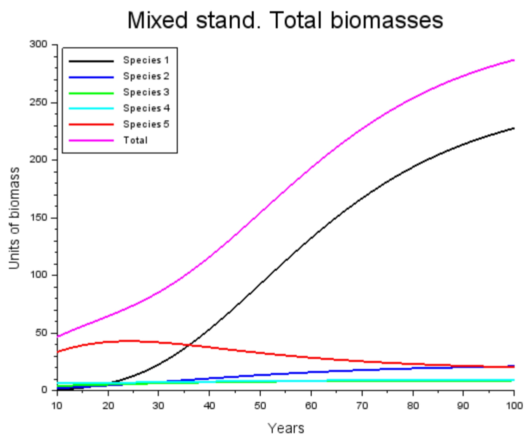
Overyielding in Forests with More Than Two Species

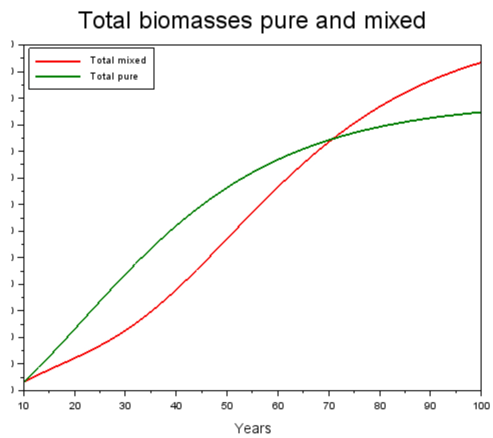
A Theoretical and Simulative Inquiry

Luís Soares Barreto









Costa da Caparica
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A table with all my texts available in the library of the Instituto Superior de Agronomia, is located in the following URL:

https://www.repository.utl.pt/handle/10400.5/196/simplesearch?filterquery=Barreto%2C+Lu%C3%ADs+Soares&filtername=author&filtertype=equals

Seventeen papers I published in *Silva Lusitana*, from 2001 to 2013, are available in thhe following URL:

http://www.scielo.mec.pt/cgi-bin/wxis.exe/iah/

With compliments

Overyielding in Forests with More Than Two Species

A Theoretical and Simulative Inquiry

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Abstract. The author uses his theory for mixed stands, and his model BACO2 for tree competition to replicate, using simulations, empirical data concerning overyielding in forests with two, and three species. He presents a general framework to contextualize the effects, on overyielding, caused by the introduction of a new species in a previous mixture of tree species. Associated to the framework, he also establishes two mechanisms to clarify the referred effects. He simulates, and analyses the effects triggered by the introduction of a third species in three types of mixtures with two species. The author also presents the listings of two functions in Scilab for the assessment of overyielding in forests with two, and three species.
Key words: overyielding; mixed forests; forests with three species; competitive hierarchy; global stand biomass

Sumário. O autor recorre à sua teoria para os povoamentos mistos e ao seu modelo BACO2 para simular dados empíricos de sobreprodução em florestas com duas e três espécies. Apresesenta um quadro geral de referência onde contextuar os efeitos na sobreprodução causados pela introdução de uma espécie numa prévia mistura de espécies arbóreas. Associadas ao quadro de referência articula dois mecanismos que clarificam os mencionados efeitos. Simula e analisa os efeitos na sobreprodução causados pela introdução de uma terceira espécie em três tipos de misturas de duas espécies. O autor também apresenta as listagens de duas funções Scilab, para a avaliação de sobreprodução em florestas com duas e três espécies.
Palavras-chave: sobreprodução; florestas mistas; florestas com três espécies; biomassa global; hierarquia competitiva

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1. Introduction

This text is a follow-up of Barreto (2020). To avoid excessive repetitions I suggest its reading and also the familiarization with my theory for tree competition, and related models, displayed in Barreto (2011: chapters 9, 14-18).

In Barreto (2020), I approached almost exclusively overyielding (**OY**) in mixed forests with two species, the type of mixed forests to whom the largest part of the literature is dedicated. Comparatively, texts dedicated to the analysis of OY in temperate forests with more than two species are scarce. In the present study, I attempt to fill the mentioned gap in my previous work, and I will deal with mixed forests with more than two species.

Now, I mainly address the following topics:

- The establishment of a general framework to contextualize the effects of adding a new a species to a previous mixture;
- The formulation of two related mechanisms to clarify the effects caused by the mentioned introduction;
- The simulative replication of OY measured in forests with two, and three species;
- The prediction of the effect on OY caused by the introduction of a third species in a mixture of two species;
- The presentation of the listings of two functions in Scilab for the assessment of overyielding in forests with two, and three species..

2. Characterizing some species

As previously, I characterize the competitive ability of a species using the ratio of *initial density/final density* in the Gompertz equation of a cohort of trees R_{-2}. The competitive coefficient in the Gompertz equation is c, see equation (4.45) in Barreto (2011). The characterization of the bionomic strategies of the tree species is approached in Barreto (2011: chapter 9). The greater is the value of R_{-2} of a species, the greater is its competitive ability.

Table 1. Characteristics of some tree species. The age of the initial density in R_{-2} is 10 years

Species	c	R_{-2}	LHS	Acronym
Abies alba	0.0310	167.4859	K-2	Aal
Abies grandis	0.0550	371.2490	K-3	Agr
Abies procera	0.0490	542.0928	K-3	Apr
Acer pseudoplatanus	0.0450	45.4938	r↔K	Aps
Alnus glutinosa	0.0290	01-11-51	r-3	Agl
Alnus rubra	0.0490	119.7037	r-1	Aru
Betula pendula	0.0350	20.0478	r-3	Bpe
Cedrus atlantica	0.0360	67.2634	-	Cat
Cryptomeria japonica	0.0710	74.1245	-	Cja
Eucalyptus grandis	0.0620	103.6363	-	Egr
Fagus sylvatica	0.0430	946.7456	K-3	Fsy
Fraxinus excelsior	0.0380	87.7699	r↔K	Fex
Larix decidua	0.0430	40.0983	K-1	Lde
Larix kaempferi	0.0350	26.9133	-	Lka
Picea abies	0.0420	210.0399	K-2	Pab
Picea mariana	0.0350	96,3245	r-1	Pma
Picea sitchensis	0.0480	72.3078	K-1	Psi
Pinus contorta	0.0380	60.0460	r-2	Pco
Pinus elliottii	0.0830	52.2975	-	Pel
Pinus halepensis	0.0820	5,4260	r-3	Pha
Pinus nigra ssp. laricio	0.0510	81.4408	K-2	Pni
Pinus pinaster	0.0500	01-06-91	r-3	Ppa
Pinus pinea	0.1050	01-11-52	K-2	Ppe
Pinus resinosa	0.0260	51,8765	r-2	Pre
Pinus strobus	0.0300	22.4103	r-3	Pst
Pinus sylvestris	0.0300	34.2586	K-1	Psy
Pinus taeda	0.0600	22.5832	r-2	Pta
Pseudotsuga menziesii	0.0460	82.1957	K-1	Pme
Quercus robur/petraea	0.0410	125.9635	K-2	Qro/Qpe
Tsuga heterophylla	0.0390	81.02	K-1	The

3. Replicating Overyielding Measured in Forests with Tree Species

Generalizing the convention adopted in Barreto (2020), a mixture of n species is referred as species 1 + species 2 + …. + species n. The ordering of the species mirror the decreasing order of their values of R_{-2}. This is, it is verified $R_{-21} > R_{-22} > .... > R_{-2n}$. A mixture with European beech, Norway spruce, and and sessile oak is named as Fsy + Pab + Qpe, because it happens 946.7>210.0 >125.9. The order of the species reflects their decreasing competitive abilities.

In a mixture with n species, initially with a total number of T trees, and an initial fraction of the dominant species equal to Pi, the initial numbers of the trees of each other species are equal to (T-Pi *T)/(n-1). We use T=10000.

Only species 1 has variable relative size of the final dimension of the tree. The relative dimension of the final size of the trees of the other n-1 species are equal to 1.

I had described the **standard mixture** here simulated. When used, changes to this structure will be described.

Pretzsch (2018:132-134) refers to 20% OY in a mixed forests with Fsy+Pab+Aal, and 43% of OY in forests with Fsy+Qpe+Psy. I show that my simulations of OY in these forests generate values compatible with these measurements.

I use both RY, and Ry as acronyms for relative yield.

The greater is the value of $R_{-2}$of a species, the greater is its competitive ability.

RY is the same when calculated from stem volume or total tree biomass.

3.1. Forests with Fsy+Pab+Aal

As usual (Barreto, 2020), I present the graphic of the relative mortality rates (**rmr**), and the relative growth rates (**rgr**) of the total biomass of the tree. Species 1 is the dominant species, in this case Fsy. Now, I apply my model BACO2 for tree competition to simulated mixed stands of the two mixtures. The simulations generate a matrix of 90 values of relative yield displayed as a surface in figures 1, and 2. Rt is the relative size of the final biomass of the tree of Fsy, and Pi the initial fraction of trees of Fsy, in the mixture. RY is the relative yielding of the mixture. RY>1 indicates OY. For more details see Barreto (2020).

Copied from the output of the Scilab script, I disclose the descriptive statistics of the 90 values of RY, represented in figure 2 (minimum, median, mean, maximum, standard deviation):

!min	median	**mean**	max	st dev !
0.701	1.214	**1.225**	1.709	0.251

The mean OY of the simulations is 22.5%, in clear agreement with the empirical data (20%). In figure 2, we see that the values of OY vary from a figure little greater than 0, (2.2%, for Pi=0.8, and Rt=1.5), to 70.9% (Pi=0.4, Rt=5). For Pi=0.8, and Rt=3, the over yielding is 20.8 %.

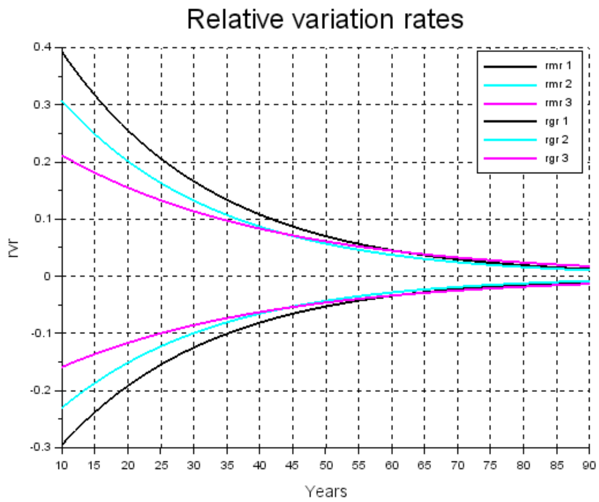


Figure 1. Graphic of the relative growth rates of total biomass of the tree, and relative rates of mortality in mixture Fsy (1) +Pab (2) + Aal (3)

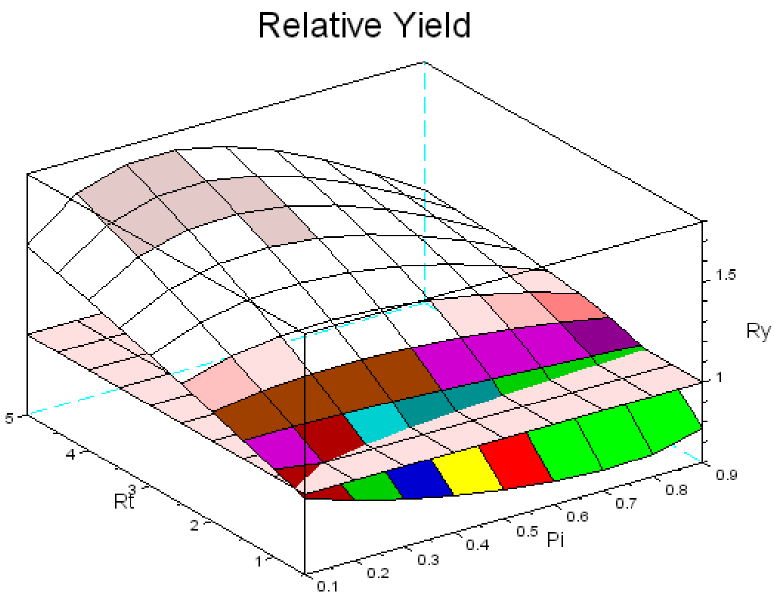


Figure 2. Simulated representation of the emergence of overyielding in mixed stands with European beech, Norway spruce, and silver fir

3.2. Forests with Fsy+Qpe+Psy

The results of the simulations for the mixture Fsy+Qpe+Psy are presented in figures 3, and 4. Again, copied from the output of the Scilab script, I disclose the descriptive statistics of the 90 values of RY, represented in figure 2:

!min	median	mean	max	st dev !
0.455	1.176	1.155	1.81	0.350

In figure 4, we see that the values of OY vary from a figure little greater than 0, (3.9%, for Pi=0.9, and Rt=2.5), to 81% (Pi=0.4, Rt=5). For Pi=0.3, and Rt=3.5, the over yielding is 43.8 %.

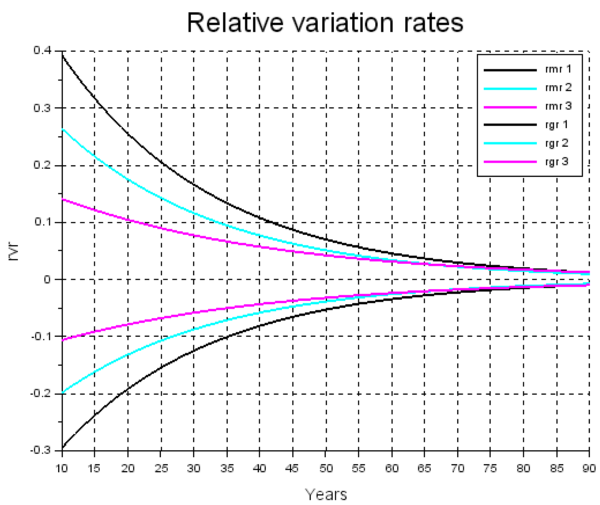


Figure 3. Graphic of the relative growth rates of total biomass of the tree, and relative rates of mortality in mixture Fsy (1) +Qpe (2) + Psy (3)

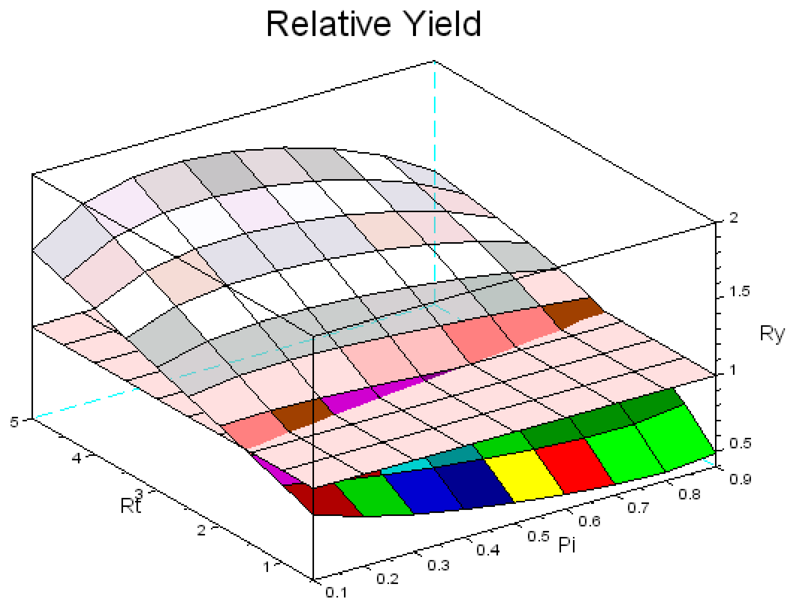


Figure 4. Simulated representation of the emergence of overyielding in mixed stands with European beech, sessile oak , and Scots pine

4. Predicting the Effect of Adding a Third Species to a Mixture

4.1. Previously Establishing Required Credibility

An important attribute of science is its predictive capacity. It is this attribute that allows the existence of technology.

My unified theory for forests (Barreto, 2010, 2011) let me anticipate the effect on OY caused by the introduction of a third species in a mixed forest with two species.

I start by retrieving a few concepts of my theory for tree competition (Barreto, 2011: chapters 14-19). Let R_{-2i} be the ratio of *initial density/final density* in the Gompertz equation of a cohort of trees of species i. The greater is R_{-2} of a species, the greater is its competitive ability. Given two species, i and j, if $R_{-2i}>R_{-2j}$, then in a mixed forest with trees of these two species, species i is the dominant species, and transfers to species j a fraction of its mortality in the process of self-thinning. We can also write $R_{-2i}>R_{-2j}$ implies the competitive hierarchy sp. i>sp. j.

Thus, the greater is the ratio R_{-2i}/R_{-2j}, the greater is the transference of mortality to the dominated species, and the greater is the possiblity of the emergence of OY, because the dominant species has also the greater relative growth rate of stem volume, and total biomass of the tree (Barreto, 2020).

These statements have empirical support. Pretzsch (2018:136,138) writes:

"Combinations of less complementary species (e.g. Norway spruce and European beech, Norway spruce and silver fir) result in lower OY than complementary species combinations

(e.g. Scots pine and European beech, European larch (*Larix decidua* MILL .) and European beech)."

The values of R_{-2} for European beech, Norway spruce , silver fir, European larch, and Scots pine, are respectively 946.7, 210.0, 167.5, 40.1, 34.2, thus the ratios R_{-2i}/R_{-2j} are smaller for mixed stands with the first two mixtures (946.7/210.0=**4.5**, 210.0/167.5=**1.5**) than with the last two mixtures (946.7/34.2= **27.7**, 946.7/40.1=**23.6**). As complementary is a relative concept, but the specific value of R_{-2} is an intrinsic parameter of the bionomic strategy of a given species, I prefer the latter characterization of the species.

European beech is present in the two groups of mixed forests. Thus, I will simulate over- and underyielding in two forests where European beech is in the mixture, one forest in each group. I select the mixtures Fsy+Pab, and Fsy+Psy. The graphics related to mixture Fsy+Pab are displayed in figures 5, and 6.

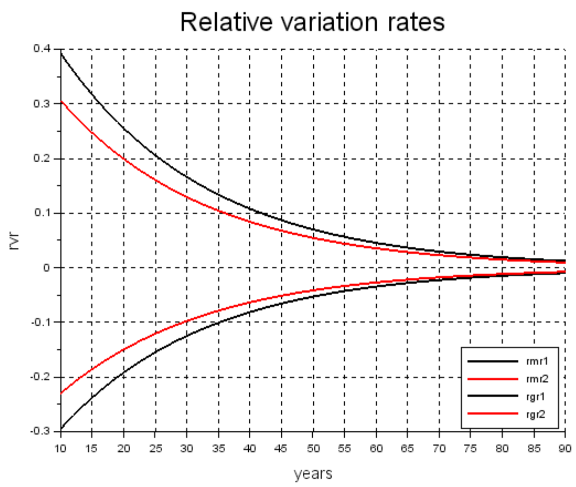


Figura 5. Graphic of the relative growth rates of total biomass of the tree, and relative rates of mortality of *Fagus sylvatica*, and *Picea abies*

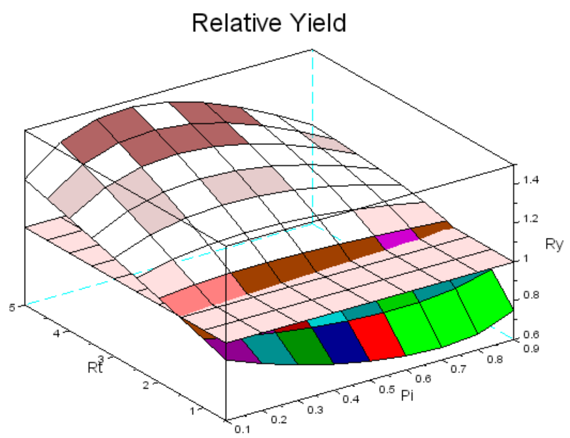


Figure 6. Simulated representation of the emergence of over- and underyielding in mixed stands with Fsy+Pab

Now, in figures 7, and 8, I show the graphics related to the second mixture, Fsy+Psy.

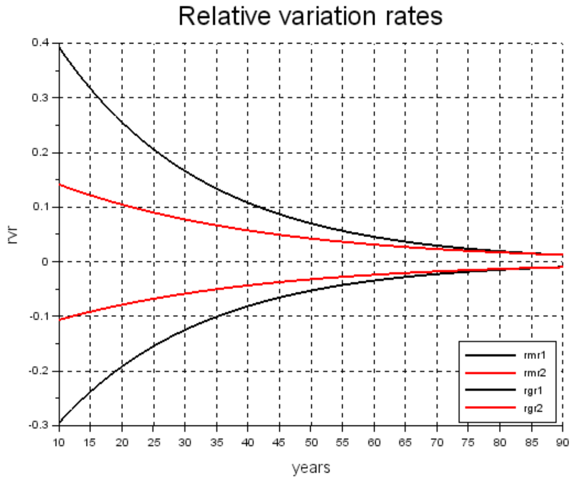


Figure 7. Graphic of the relative growth rates of total biomass of the tree, and relative rates of mortality of *Fagus sylvatica*, and *Pinus sylvestris*

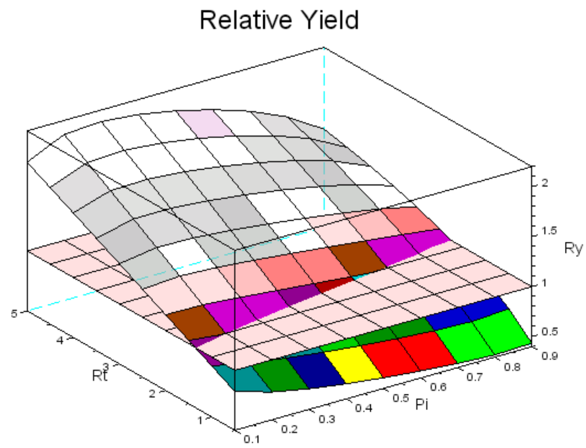


Figure 8. Simulated representation of the emergence of over- and underyielding in mixed stands with Fsy+Psy

In table 2, I disclose the descriptive statistics of the 90 values of RY of each mixture. The greater OY of the mixture Fsy+Psy is conspicuous. Also in this table, the greater dispersion of the values of RY is associated to the higher value of the ratio R_{-2i}/R_{-2j}.

Table 2. Descriptive statistics of the relative overyieldind of the two mixtures

Mixtures	R_{-2i}/R_{-2j}	Minimum	Median	Mean	Maximum	St. deviation
Fsy+Pab	**4.51**	**0.665**	**1.114**	**1.106**	**1.479**	**0.199**
Fsy+Psy	**27.68**	**0.422**	**1.267**	**1.244**	**2.084**	**0.441**

I display also the confirmatory t test, obtained in R:

```
A=c(0.913, 0.841, 0.782, 0.735, 0.698,  0.673, 0.665, 0.681, 0.752,
 0.955, 0.916, 0.885, 0.860, 0.842, 0.832, 0.834, 0.853, 0.899,
 0.995, 0.987, 0.978, 0.969, 0.962, 0.956, 0.955, 0.959, 0.972,
 1.035, 1.055, 1.065, 1.067, 1.063, 1.056, 1.045, 1.032, 1.017,
 1.074, 1.12,  1.145, 1.154, 1.151, 1.137, 1.115, 1.084, 1.046,
 1.112, 1.181, 1.218, 1.232, 1.226, 1.205, 1.171, 1.124, 1.067,
 1.149, 1.24,  1.287, 1.303, 1.293, 1.263, 1.216, 1.156, 1.083,
 1.185, 1.295, 1.351, 1.367, 1.351, 1.312, 1.254, 1.181, 1.096,
 1.221, 1.349, 1.411, 1.425, 1.403, 1.355, 1.286, 1.202, 1.105,
 1.256, 1.4,   1.467, 1.479, 1.45,  1.392, 1.314, 1.219, 1.114)

S=c(0.785, 0.677, 0.605, 0.550, 0.506, 0.470, 0.440, 0.421, 0.434,
 0.910, 0.837, 0.780, 0.733, 0.694, 0.661, 0.635, 0.622, 0.646,
 1.034, 0.995, 0.952, 0.911, 0.875, 0.842, 0.815, 0.798, 0.810,
 1.157, 1.151, 1.121, 1.086, 1.049, 1.014, 0.982, 0.954, 0.941,
 1.28,  1.306, 1.288, 1.256, 1.218, 1.178, 1.137, 1.093, 1.048,
 1.403, 1.46,  1.452, 1.422, 1.381, 1.334, 1.281, 1.218, 1.137,
 1.525, 1.612, 1.614, 1.584, 1.539, 1.483, 1.415, 1.33,  1.212,
 1.646, 1.763, 1.773, 1.743, 1.692, 1.625, 1.541, 1.432, 1.277,
 1.767, 1.913, 1.93,  1.898, 1.839, 1.76,  1.659, 1.525, 1.333,
 1.888, 2.061, 2.084, 2.049, 1.982, 1.89,  1.769, 1.61,  1.382)


t.test(A,S)

	Welch Two Sample t-test

data:  A and S
t = -2.6917, df = 123.91, p-value = 0.008091
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -0.23820292 -0.03633041
sample estimates:
mean of x mean of y 
 1.106400  1.243667
```

The results provided by the application of model BACO2, and my theory to mixed stands, are consistent with the empirical evidence displayed by Prof. Hans Pretzsch. These results, and those displayed in section 3 reassure the truthlikeness of both, theory and model.

4.2. A General Framework

The effect on OY caused by adding a third species k to a mixture with species i, and j can be deduced from the analysis of OY in mixture with two species. Thus, the this effects depends on:

- **The initial proportions of the three species;**
- **The ratios R_{-2i}/R_{-2j}, R_{-2i}/R_{-2k}, and R_{-2j}/R_{-2k};**
- **The relative sizes of the trees of the three species.**

I call this three statements the general framework. It underpins the following three conjectures, that favour the systematization of our approach to the main issue:

C1. The greater is the competitive ability of the new species, relative to the existing two species, the greater may be the increment of OY in the stand with three species compared to the one with two species. Let considered a stand with two species i, j to which we introduce species k. This is, if $R_{-2k} > R_{-2i}$, R_{-2j}, then the introduction of species k may increase OY. Thus, if the new species is a K-3 (e.g., *Fagus sylvatica*) the RY of the new stand may be greater than the one of the two species.

C2. The smaller is the competitive ability of the new species, relative to the existing two species, the greater may be the increment of OY in the stand with three species compared to the one with two species. Let considered a stand with two species i, j to which we introduce species m. If $R_{-2m} < R_{-2i}$, R_{-2j}, then the introduction of species m may increase the OY, because both initial two species transfer to it part of their self-thinning mortality. Thus, if the new species is a r-3 (e.g., *Pinus pinaster*) the RY of the new stand may be greater than the one of the two species.

C3. If to a stand with two species with close competitive ability, we add a new species with comparable competitive ability, the OY evinces little change. An example is a stand with *Pseudotsuga menziesii* and *Picea sitchensis* to which we introduce *Tsuga heterophylla*.

I did not expressed definitive and strong statements on C1, C2, and C3 because the general framework does not support them.

In all three situations, the greater is the relative size of the trees of the third species the greater is its effect on OY. The relative sizes of the trees of each species mirror the effect of the site quality on the species.

This line of thought can be extended to an initial forest previously with any number of trees.

Now, we introduce some simulative examples that illustrate the main predictions here stated.

4.3. The Mixture Apr+Agr

Let us consider a mixture with two species with Ry equal to about one (RY≈1), because the two species have very close competitive ability. Thus, adding a third species is formally equal to form a mixture with two species. This is the case of a mixture Apr+Agr.

The competitive hierarchy of the species used in this subsection is:

Fsy>Apr>Agr>Bpe

I start by introducing the graphic of the mixture with two species. In figure 9.

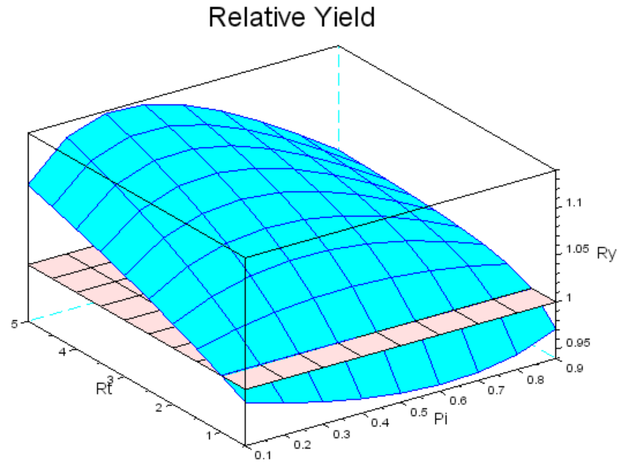


Figure 9. Simulated representation of the emergence of over- and underyielding in mixed stands with Apr+Agr

To the previous figure, I had the surface of a standard mixture with Fsy, as displayed in figure 10. The new mixture, as expected has higher values of Ry, but for values of Rt<1. The mean of the 90 values of Ry in the mixture with two species is 1.050, and in the mixture with three species 1.102. The relative smaller increase is because the two initial species are also K-3 strategists.

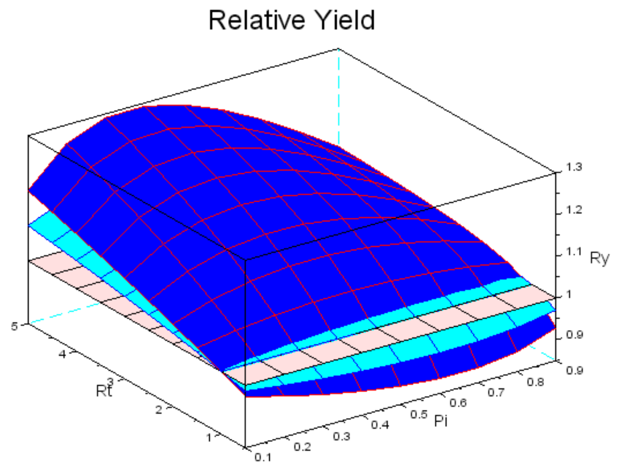


Figure 10. Simulated representation of the emergence of over- and underyielding in mixed stands with Apr+Agr, and *Fsy+* Apr+Agr

Now, we introduce a r-3 strategist. I select Bpe. The result of the simulation of the emergence of OY, with the standard mixture is displayed in figure 11.

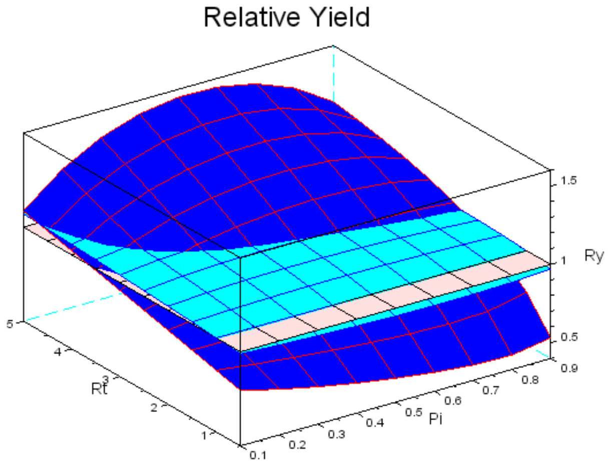


Figure 11. Simulated representation of the emergence of over- and underyielding in mixed stands with Apr+Agr, and Apr+Agr+Bpe

In figure 11, for certain combination of Rt, and Pi, Ry is greater in the mixture with three species than in the mixture with two species. Also, underyielding is more acute in the mixture with thee species.

To illustrate the effct of the relative sizes of the initial populations we use the following proportions Pi, 0.2*(1-Pi), 0.8*(1-Pi). This is, we increase the initial proportion of the trees of the introduced species, relative to the standard mixture (Pi, 0.5*(1-Pi), 0.5*(1-Pi)), As shown in figure 12, the combinations of Rt, and Pi with greater Ry are more numerous, with the increasing of proportion of the weaker competitor.

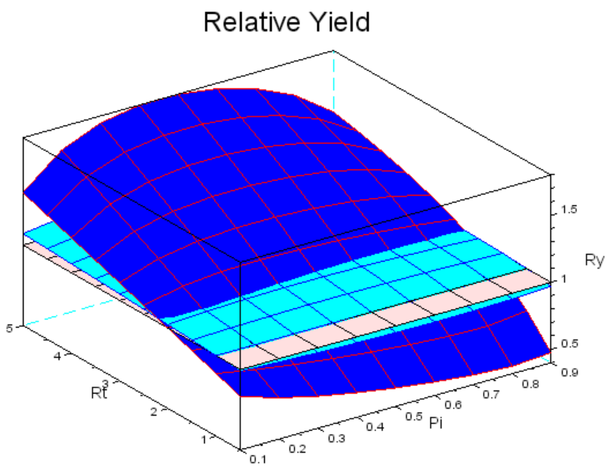


Figure 12. Simulated representation of the emergence of over- and underyielding in mixed stands with Apr+Agr, and Apr+Agr+Bpe, with greater initial proportion of Bpe than in the standard mixture

In table 3, we display the descriptive statistics of the the three mixtures simulated.

Table 3. Descriptive statistics of the relative over- and underyieldinds of the three mixtures simulated in figures 10, 11, and 12

Mixtures	Minimum	Median	Mean	Maximum	St. deviation
Apr+Agr	0.947	1.053	1.050	1.137	0.048
Fsy+Apr+Agr	0.885	1.107	1.102	1.292	0.102
Apr+Agr+Bpe	0.491	1.054	1.040	1.550	0.272
Apr+Agr+Bpe Non standard mixture	0.455	1.155	1.148	1.799	0.350

Now, we are in a position that allow us, tentatively, to establish the two mechanisms that clarify the behaviour of OY in the graphics of figures 10-12.

M1.The impact of adding a third dominant species (e.g., Fsy) to a mixture with two species is the balance between the increasing in production due to the newcomer, and the loss of production inflicted in the original species, due to its initial smaller proportions, and greater mortality. For a given initial proportions of the three species, there is a threshold value of the final size of the trees of the dominant species that must be surpassed to occur OY.

M2.The impact of adding a third dominated species (e.g., Bpe) to a mixture with two species is the balance between the increasing in production due to the newcomer (lees mortality of the initial more productive species), and the loss of production inflicted in the original species, due to decreasing the proportions of the two more productive species. For a given initial proportions of the three species, there a threshold value of the final size of the trees of the dominant species that must be surpassed to emerge OY.

4.4. The Mixture Pab+Psy

Now we approach a most general case when, in the initial mixture, species 1 has an acute dominance over species 2. We select the mixture Pab+Psy.

The graphic of OY in the mixture with two species is exhibited in figure 13.

The competitive hierarchy of the species used in this subsection is:

Fsy>Pab>Psy>Bpe>Ppa

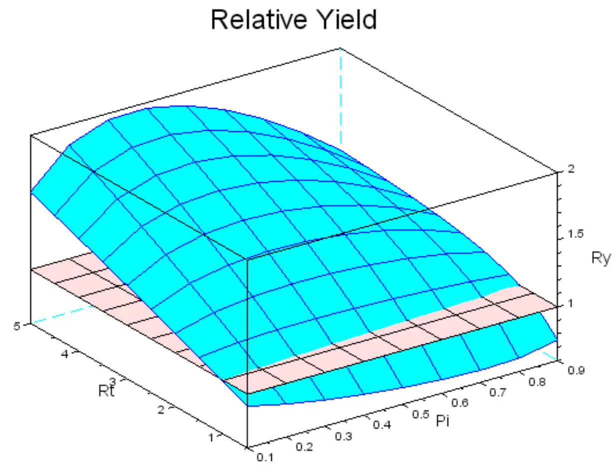


Figure 13. Simulated representation of the emergence of over- and underyielding in mixed stands with Pab+Psy

Now, we add to the initial mixture Fsy, and apply the standard mixture. The respective graphic is displayed in figure 14. The increasing of relative yield occurs only for extreme values of Pi, and Rt.

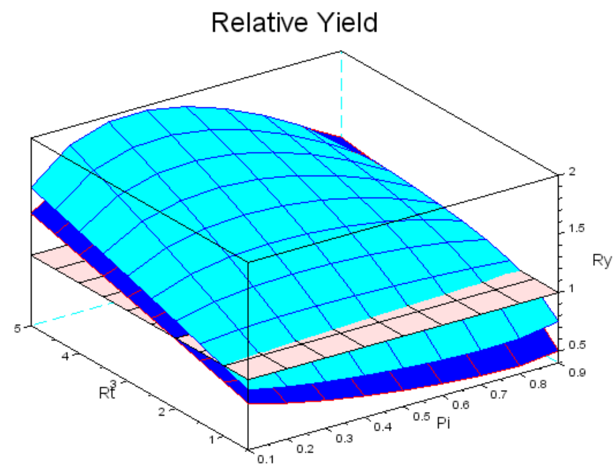


Figure 14. Simulated representation of the emergence of over- and underyielding in mixed stands with Pab+Psy, and Fsy+Pab+Psy, with standard mixture

The simulation of the same mixture with the non standard mixture already described is presented in figure 15. The increasing of the initial proportion of the weaker competitor generated more combinations of Rt, and Pi with higher Ry than in the mixture with two species.

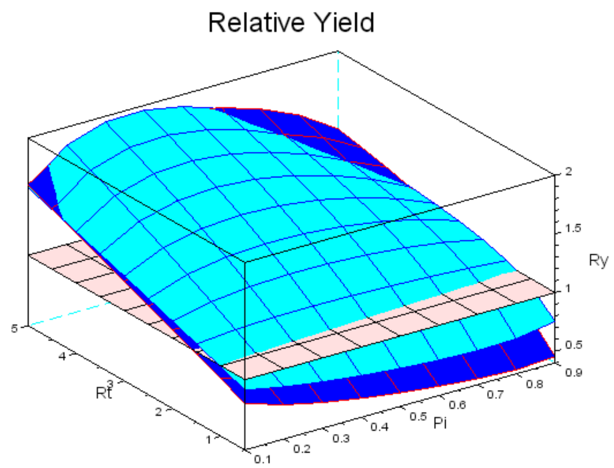


Figure 15. Simulated representation of the emergence of over- and underyielding in mixed stands with Pab+Psy, and Fsy+Pab+Psy, with non standard mixture

Let us scrutinize the effect of introducing a dominated species in the initial mixture. We select the non standard mixture, and species Bpe, and Ppa. The respective surfaces are displayed in figures 15, and 16.

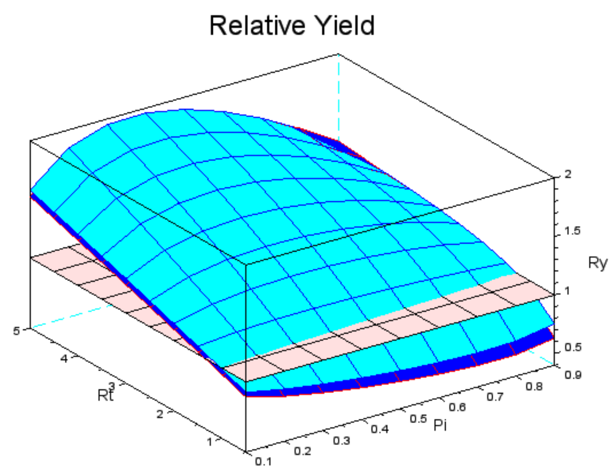


Figure 16. Simulated representation of the emergence of over- and underyielding in mixed stands with Pab+Psy, and Pab+Psy+Bpe, with non standard mixture

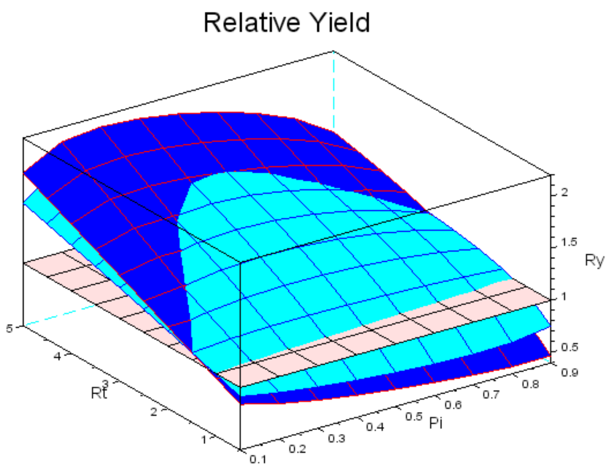


Figure 17. Simulated representation of the emergence of over- and underyielding in mixed stands with Pab+Psy, and Pab+Psy+Ppa, with non standard mixture

In table 4 we display the descriptive statistics of the five simulations we did.

Table 4. Descriptive statistics of the relative over- and underyieldinds of the three mixtures simulated in figures 13 to 17

Mixtures	Minimum	Median	Mean	Maximum	St. deviation
Pab+Psy	0.691	1.300	1.314	1.959	0.326
Fsy+Pab+Psy	0.465	1.130	1.117	1.719	0.323
Fsy+Pab+Psy Non standard mixture	0.434	1.197	1.182	1.901	0.386
Pab+Psy+Bpe Non standard mixture	0.582	1.242	1.230	1.866	0.330
Pab+Psy+Ppa Non standard mixture	0.466	1.297	1.279	2.098	0.431

For the combinations of values of Rt, and Pi we used, in the majority of the new stands with three species, the losses due to the introduction of the third species were greater than the gains. The more favourable results were obtained with the introduction of a weaker competitor (Bpe, Ppa).

Let us be more detailed. In table 5, we display the ratios of R_{-2} of Fsy, and other for initial species, and the ratio of the relative growth rate of the total biomass of the species ($R_{0.66666}$). Now, the smaller is the ratio $R_{0.6i}/R_{0.6j}$ the greater is the growth of total biomass of species i compared to the species j.

Table 5. Ratios R_{-2i}/R_{-2j} e $R_{0.6i}/R_{0.6j}$ relative to the initial species, and the introduced third species, in the mixtures considered in this subsection, and the previous one

Species	R_{-2i}/R_{-2j}	$R_{0.6i}/R_{0.6j}$
Fsy/Apr	1.746	0.830
Fsy/Agr	2.550	0.732
Fsy/Pab	4.507	0.605
Fsy/Psy	27.634	0.331
Apr/Bpe	27.040	0.333
Agr/Bpe	18.518	0.378
Pab/Bpe	10.477	0.457
Psy/Bpe	1.709	0.836
Pab/Ppa	34.895	0.306
Psy/Ppa	5.692	0.560

Table 5 shows that in the initial mixture Apr+Agr the ratios R_{-2i}/R_{-2j} (1.7, 2.5) are much smaller than in the mixture Pab+Fsy (4.5, 27.6), thus in the former mixture the increase of mortality due to the introduction of the third species is smaller. In the mixture Apr+Agr the loss of production of Apr, and Apr is over compensated by the higher growth rate of total biomass and less mortality of Fsy, for values of Rt>1. This is virtually absent with the standard mixture in forests Fsy+Pab+Psy. The relative greater production of Fsy, and its less mortality is not sufficient to overcompensate the loss of production of Pab, and Psy. The use of the non standard mixture allows a greater transference of mortality to Psy, and the new mixture has higher Ry for more combinations of high values of Rt, and Pi.

In forests with Apr+Agr+Bpe the high values of R_{-2i}/R_{-2j} (27.0, 18.5) allows a great transference of mortality from Apr, and Agr to the less productive species Bpe, and in mixtures Apr+Agr+Bpe higher values of Ry relative to the initial mixture (Apr+Agr) emerge for several combinations of Rt, and Pi.

In forests with Pab+Psy+Bpe the transference of mortality to Bpe is not over compensated by the over production of Pab, and Psy, because their growths rates of total biomass are not enough greater than the growth rate of biomass of Bpe.

This situation is attenuated when we replace Bpe by Ppa and higher values of R_{-2i}/R_{-2j} occur, respectively [34.9, 5.7] versus [10.4, 1.7]. There is a greater transference of mortality to the less productive species, and the growths rates of total biomass of Pab, and Psy are now enough greater than the growth rate of biomass of Ppa to cause the emergence of more comparatively numerous values of Ry greater than in the initial mixture.

4.5. The Mixture Pme+The

In this subsection we will illustrate conjecture **C3**. To a stand with two species with close competitive ability, we add a new species with comparable competitive ability. To a stand with *Pseudotsuga menziesii* and *Tsuga heterophylla* we add *Picea sitchensis* and we will see that the changes of the Ry are residual.

The competitive hierarchy here prevalent is:

Pme>The>Psi

The graphic with the Ry of the mixture with two species is exhibited in figure 18.

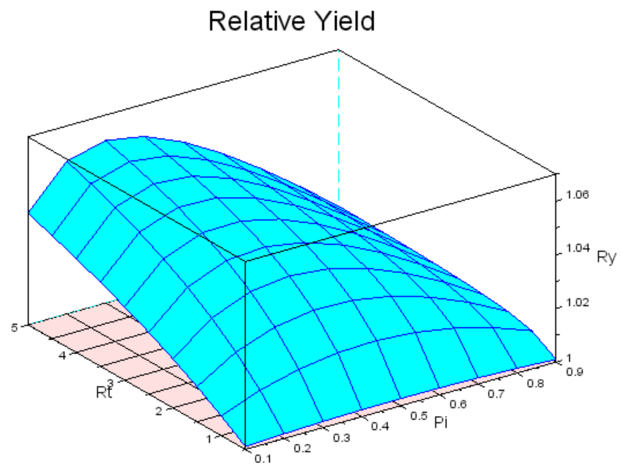


Figure 18. Simulated representation of the emergence of over- and underyielding in mixed stands with Pme+*The*

Now, in the initial mixture we introduce Psi, and apply the standard mixture. The respective graphic is displayed in figure 19. The changes in the values of relative yield are negligible, as confirmed in tables 6, and 7.

Table 6. Descriptive statistics of the relative over- and underyieldinds of the three mixtures simulated in figures 18 and 19

Mixtures	Minimum	Median	Mean	Maximum	St. deviation
Pme+The	1.001	1.029	1.029	1.060	0.017
Pme+The+Psi	0.997	1.029	1.029	1.057	0.016

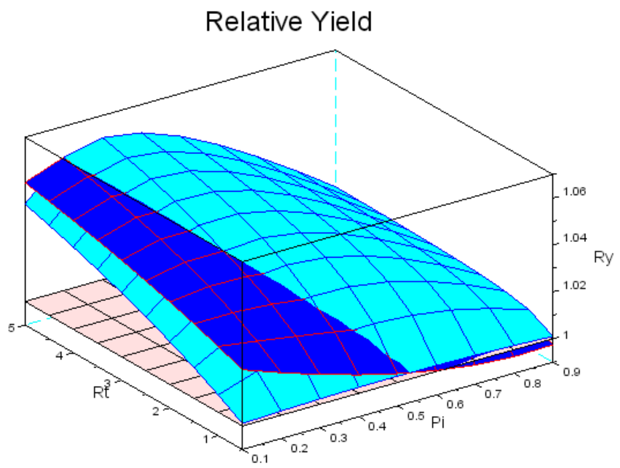


Figure 19. Simulated representation of the emergence of over- and underyielding in mixed stands with Pme+The, and Pab+The+Psi, with standard mixture

Table 6. Difference of the matrix of values of Ry of the mixture with three species minus the same matrix of the mixture with two species (Ry_{3sps}-Ry_{2sps})

Rt	Pi								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	0.023	0.016	0.010	0.005	0.001	-0.002	-0.004	-0.005	-0.004
1	0.021	0.013	0.006	0.002	-0.002	-0.004	-0.005	-0.004	-0.003
1.5	0.019	0.010	0.004	-0.001	-0.003	-0.005	-0.005	-0.004	-0.002
2	0.017	0.008	0.002	-0.002	-0.004	-0.005	-0.005	-0.004	-0.002
2.5	0.015	0.006	0.000	-0.003	-0.005	-0.005	-0.005	-0.004	-0.002
3	0.014	0.004	-0.001	-0.004	-0.005	-0.006	-0.005	-0.004	-0.002
3.5	0.013	0.003	-0.002	-0.005	-0.006	-0.006	-0.005	-0.004	-0.002
4	0.011	0.002	-0.003	-0.005	-0.006	-0.006	-0.005	-0.003	-0.002
4.5	0.010	0.001	-0.004	-0.006	-0.006	-0.006	-0.005	-0.003	-0.002
5	0.009	0.000	-0.004	-0.006	-0.006	-0.006	-0.005	-0.003	-0.002

The simulation here presented corroborated conjecture C3.

I concentrated my analysis, and conceptualization in the case of an initial forest with two species, but the general framework, and the two mechanisms can easily be extended to an initial forest with any number of species.

5. Two Simulators for the Assessment of Overyielding

In this section we introduce simulators ry2plot, and ry3plot, respectively, for the simulation of self-thinned even-aged mixed forests with two, and three species. They are Scilab functions whose inputs are:

c_i (i=1,2 and i=1,2,3) The values of the coefficient of competition in the equation of Gompertz. For some species they are exhibited in table 1;

r_i (i=1,2 and i=1,2,3) The values of **R_{-2}**. For some species they are exhibited in table 1;

p_i (i=1,2 and i=1,2,3) . They are the initial number of trees of species i;

f_i (i=1,2 and i=1,2,3) . They are the final stem volume of the average tree of species i;

g is the age of the trees in the last year of simulation minus 10.

For easier interpretations of the output of the simulations, we suggest that the species in the input to be introduced in decreasing order of R_{-2}. This order is in agreement with the competitive hierarchy.

Let us consider a forest with Fsy+Qro+Lde, an hypothetical input in simulator ry3plot is:

ry3(c_1, r_1, c_2, r_2, c_3, r_3, p_1, p_2, p_3, f_1, f_2, f_3, g)

ry3(0.043, 946.7, 0.041, 126.0, 0.043, 40.0, 2000,2000,2000, 7, 6,3,120)

If instead of stem volume of the mean tree we enter total biomass of the mean tree, in the listings of the functions the value of a must be changed from -1.5 to -1.333333. The functions mention indistinctively volume and biomass.

The written output of the functions are:

- For each species, and last year of simulation, the rations of 'standing volume in mixed forest/standing volume in pure forest';
- The value of the relative yield, Ry. As already known Ry>1 implies overyielding.

An example of the graphical output of the simulators, for a forest with five species, is presented in the cover of this text.

5.1 Simulator ry2plot

The listing of simulator ry2plot is the following:

```
//**************************************
//*     Simulator ry2plot               *
//* © Luís Soares Barreto, 2010, 2020   *
//*                                     *
//**************************************

function [R]=ry2(c1,r1,c2,r2,p1,p2,f1,f2,g)
  //it assumes the size of the tree is the stem volume
 //If the size of the tree is total biomass make a=-1.333333
  a=-1.5;
  b=-0.5;
  rv1=r1^a;
  rv2=r2^a;
  rv1t=r1^b;rv2t=r2^b;
  scf(0)
  clf(0)
  t0=0:g-10;
  x=10:g;
  //Relative mortality rates, and
  //Relative growth rates of the total biomass of a tree
  rmr1=-c1*log(r1)*exp(-c1*t0);
  rmr2=-c2*log(r2)*exp(-c2*t0);
  rgr1=-c1*log(rv1)*exp(-c1*t0);
  rgr2=-c2*log(rv2)*exp(-c2*t0);
  M=[rmr1;rmr2;rgr1;rgr2;]';
  plot2d(x,[M])
   legend(["rmr 1","rmr 2","rgr 1","rgr 2"], a=1)
  xtitle("Relative variation rates","Years","rvr")
```

```
b=get("current_axes");
b.title.font_size=5;
b.x_label.font_size=3;
b.y_label.font_size=3;
b.z_label.font_size=3;
b.children // list the children of the axes.
poly1= b.children.children;
poly1.thickness = [2,2,2,2];
xgrid()

//pure stands biomasses
k=0:g;
vf1=f1;vf2=f2;
p1f=p1/r1;p2f=p2/r2;
b1f=p1f*vf1;b2f=p2f*vf2;
bpt1=b1f*rv1t^exp(-c1*k);
bpt2=b2f*rv2t^exp(-c2*k);

//BACO2
f1='y(1)/(y(1)+y(2))';
f2='y(2)/(y(1)+y(2))';
rmr1='-c1*log(r1)*exp(-c1*t)';
rmr2='-c2*log(r2)*exp(-c2*t)';
deff("yprim=f(t,y)",[
"yprim1=y(1)*evstr(rmr1)*(1+evstr(f2)*log((evstr(rmr2))/(evstr(rmr1))))";..
"yprim2=y(2)*evstr(rmr2)*(1+evstr(f1)*log((evstr(rmr1))/(evstr(rmr2))))";..
"yprim=[yprim1;yprim2]"])
y0=[p1,p2];
t0=0;
t=0:g;
[M]=(matrix(ode(y0,t0,t,f),2,g+1))';
```

```
//biomasses of the trees
 a1=[vf1*rv1^exp(-c1*t)];
  a2=[vf2*rv2^exp(-c2*t)];
  //Pure forest standing volume at age g
  vp1=bpt1(1,g);
  vp2=bpt2(1,g);
   Tp=vp1+vp2;

  //Biomasses of the mixed stand
  vm1=M(g+1,1)*a1(1,g+1);
  vm2=M(g+1,2)*a2(1,g+1);
 //Standing biomasses all time
vm1t=M(:,1).*a1(1,:)';
vm2t=M(:,2).*a2(1,:)';
 tot=vm1t'+vm2t';
prj=matrix([vm1t vm2t tot'],g+1,3);
scf(1)
clf(1)
t=10:g+10;
plot2d(t,prj)
xtitle("Mixed stand. Total biomasses","Years","Units of biomass")
  b=get("current_axes");
  b.title.font_size=5;
  b.x_label.font_size=3;
  b.y_label.font_size=3;
  //b.z_label.font_size=3;
   b.children // list the children of the axes.
  poly1= b.children.children; //store polyline handle into poly1
  //poly1.foreground = [1,2,5]; // another way to change the style...
  poly1.thickness = [4, 2,2];
  xgrid()
```

```
legend(["Species 1","Species 2","Total"], a=2)
 //mixed/(sum of pure stands)
 R=(vm1+vm2)/Tp;
disp('Ratio mixed/pure for the species')
 disp([vm1/vp1 vm2/vp2])
 tit=['Over or underyieldig'];
 disp(tit)
 format(6)
TP=bpt1+bpt2;
 G=[tot' TP'];
 scf(2)
 clf(2)
 plot(t,G)
 xtitle("Total biomasses pure and mixed","Years","Units of biomass")
 b=get("current_axes");
 b.title.font_size=5;
 b.x_label.font_size=3;
 b.y_label.font_size=3;
 b.children // list the children of the axes.
 poly1= b.children.children; //store polyline handle into poly1
 poly1.foreground = [1,5]; // another way to change the style...
 poly1.thickness = [3,2];
 xgrid()
legend(["Total mixed","Total pure"], a=2)
disp("Over- or underyielding")
endfunction
```

As usual, after running the function we introduce the input. Let us consider a forest with Qro+Lde.

--> ry2(0.041,125.963,0.043,40.008,1000,1000,5.7,3.3,90)

"Ratio mixed/pure for the species"

1.8808055   0.6194312

"Over or underyieldig"

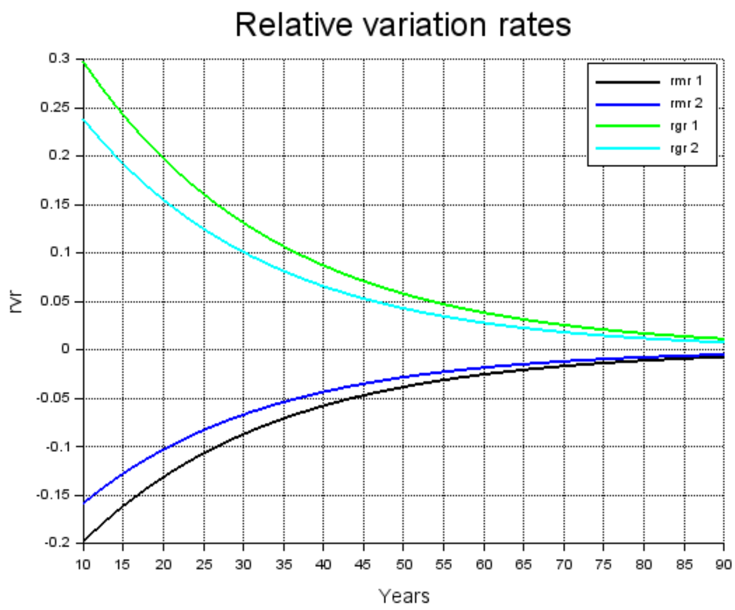
WARNING: Transposing row vector X to get compatible dimensions

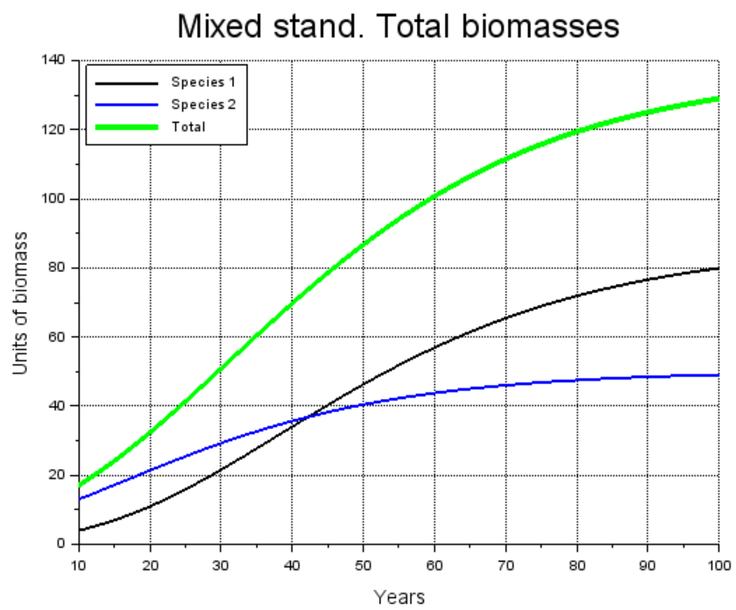
"Over- or underyielding"

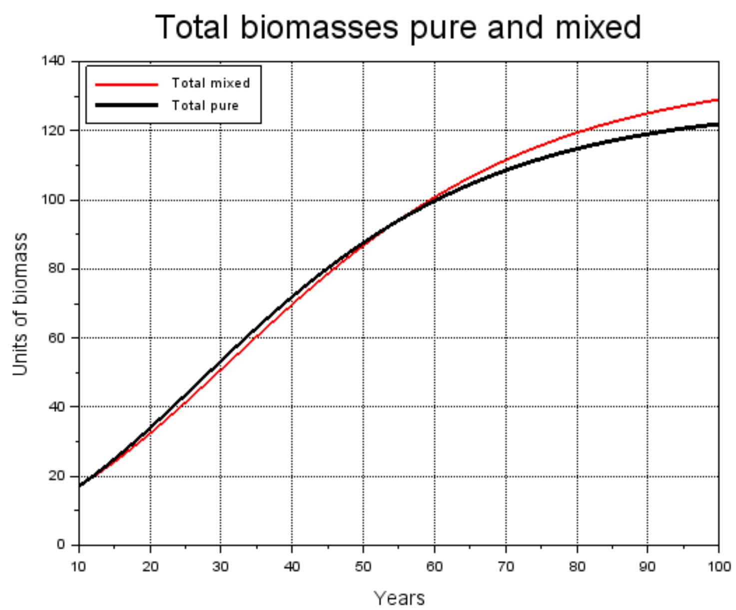
ans =

1.060   ← **A very small amount of overyielding occurs as evinced in the third graphic**

We also obtain the following three graphics:




Mixed stand. Total biomasses
Species 1
Species 2
Total
Units of biomass
Years
0
20
40
60
80
100
120
140
10
20
30
40
50
60
70
80
90
100


Total biomasses pure and mixed
Total mixed
Total pure
Units of biomass
Years
0
20
40
60
80
100
120
140
10
20
30
40
50
60
70
80
90
100

5.2 Simulator ry3plot

The listing of simulator ry3plot is the following:

```
//*************************************
//*     Simulator ry3plot                *
//* © Luís Soares Barreto, 2010, 2020    *
//*                                      *
//*************************************

function [R]=ry3(c1, r1, c2, r2, c3, r3, p1, p2, p3, f1, f2, f3, g)


 //it assumes the size of the tree is the stem volume
 //If the size of the tree is total biomass make a=-1.333333
  a=-1.5;
  b=-0.5;
  rv1=r1^a;
  rv2=r2^a;
  rv3=r3^a;
  rv1t=r1^b;rv2t=r2^b;rv3t=r3^b;

  scf(0)
  clf(0)
  t0=0:g-10;
  x=10:g;

  //Relative mortality rates, and
 //Relative growth rates of the total biomass of a tree
  rmr1=-c1*log(r1)*exp(-c1*t0);
  rmr2=-c2*log(r2)*exp(-c2*t0);
  rmr3=-c3*log(r3)*exp(-c3*t0);
  rgr1=-c1*log(rv1)*exp(-c1*t0);
  rgr2=-c2*log(rv2)*exp(-c2*t0);
  rgr3=-c3*log(rv3)*exp(-c3*t0);
  M=[rmr1;rmr2;rmr3;rgr1;rgr2;rgr3]';


  plot2d(x,[M])
   legend(["rmr 1","rmr 2","rmr 3","rgr 1","rgr 2","rgr 3"], a=1)
  xtitle("Relative variation rates","Years","rvr")
  b=get("current_axes");
  b.title.font_size=5;
  b.x_label.font_size=3;
  b.y_label.font_size=3;
  b.z_label.font_size=3;
  b.children // list the children of the axes.
  poly1= b.children.children;
  poly1.thickness = [2,2,2,2,2,2];
  xgrid()

 //pure stands biomasses
  k=0:g;
  vf1=f1;vf2=f2;vf3=f3;
```

```
 p1f=p1/r1;p2f=p2/r2;p3f=p3/r3;
 b1f=p1f*vf1;b2f=p2f*vf2;b3f=p3f*vf3;
 bpt1=b1f*rv1t^exp(-c1*k);
 bpt2=b2f*rv2t^exp(-c2*k);
    bpt3=b3f*rv3t^exp(-c3*k);

//BACO2

 f1='y(1)/(y(1)+y(2)+y(3))';
 f2='y(2)/(y(1)+y(2)+y(3))';
 f3='y(3)/(y(1)+y(2)+y(3))';

 rmr1='-c1*log(r1)*exp(-c1*t)';
 rmr2='-c2*log(r2)*exp(-c2*t)';
 rmr3='-c3*log(r3)*exp(-c3*t)';

 deff("yprim=f(t,y)",[
 "yprim1=y(1)*evstr(rmr1)*(1+evstr(f2)*log((evstr(rmr2))/(evstr(rmr1)))+evstr(f3)*log((evstr(rmr3))/
(evstr(rmr1))))";..
 "yprim2=y(2)*evstr(rmr2)*(1+evstr(f1)*log((evstr(rmr1))/(evstr(rmr2)))+evstr(f3)*log((evstr(rmr3))/
(evstr(rmr2))))";..
 "yprim3=y(3)*evstr(rmr3)*(1+evstr(f1)*log((evstr(rmr1))/(evstr(rmr3)))+evstr(f2)*log((evstr(rmr2))/
(evstr(rmr3))))";..
 "yprim=[yprim1;yprim2;yprim3]"])
 y0=[p1,p2,p3];
 t0=0;
 t=0:g;

 [M]=(matrix(ode(y0,t0,t,f),3,g+1))';

 //biomasses of the trees


 a1=[vf1*rv1^exp(-c1*t)];
 a2=[vf2*rv2^exp(-c2*t)];
 a3=[vf3*rv3^exp(-c3*t)];
 //volume em pé puro aos 90 anos
 vp1=bpt1(1,g);
 vp2=bpt2(1,g);
 vp3=bpt3(1,g);
 Tp=vp1+vp2+vp3;

 //Biomasses of the mixed stand
 vm1=M(g+1,1)*a1(1,g+1);
 vm2=M(g+1,2)*a2(1,g+1);
 vm3=M(g+1,3)*a3(1,g+1);


//Standing biomasses all time
vm1t=M(:,1).*a1(1,:)';
vm2t=M(:,2).*a2(1,:)';
 vm3t=M(:,3).*a3(1,:)';

tot=vm1t'+vm2t'+vm3t';
```

```
proj=matrix([vm1t vm2t vm3t tot'],g+1,4);
scf(1)
clf(1)
t=10:g+10;
plot2d(t,proj)
xtitle("Mixed stand. Total biomasses","Years","Units of biomass")
 b=get("current_axes");
 b.title.font_size=5;
 b.x_label.font_size=3;
 b.y_label.font_size=3;
 //b.z_label.font_size=3;
  b.children // list the children of the axes.
 poly1= b.children.children; //store polyline handle into poly1
 //poly1.foreground = [1,2,4,5]; // another way to change the style...
 poly1.thickness = [4, 2,2,2];
legend(["Species 1","Species 2","Species 3","Total"], a=2)
 xgrid()

 //mixed/(sum of pure stands)
 R=(vm1+vm2+vm3)/Tp;

 disp('Ratio mixed/pure for the species')
 disp([vm1/vp1 vm2/vp2 vm3/vp3])

 tit=['Over or underyieldig'];
 disp(tit)
 format(6)

 TP=bpt1+bpt2+bpt3;
 G=[tot' TP'];
 scf(2)
 clf(2)
 plot(t,G)
  xtitle("Total biomasses pure and mixed","Years","Units of biomass")
 b=get("current_axes");
 b.title.font_size=5;
 b.x_label.font_size=3;
 b.y_label.font_size=3;
 //b.z_label.font_size=3;
  b.children // list the children of the axes.
 poly1= b.children.children; //store polyline handle into poly1
 poly1.foreground = [1,5]; // another way to change the style...
 poly1.thickness = [2,2];
 xgrid()
legend(["Total mixed","Total pure"], a=2)

 disp("Over- or underyielding")
endfunction
```

After running the function we introduce the input. Let us consider a forest with Fsy+Qro+Lde.

--> ry3(0.043,946.746,0.041,125.963,0.043,40.008,1000,1000,1000,6,5,3,90)

"Ratio mixed/pure for the species"

8.351   0.843   0.335

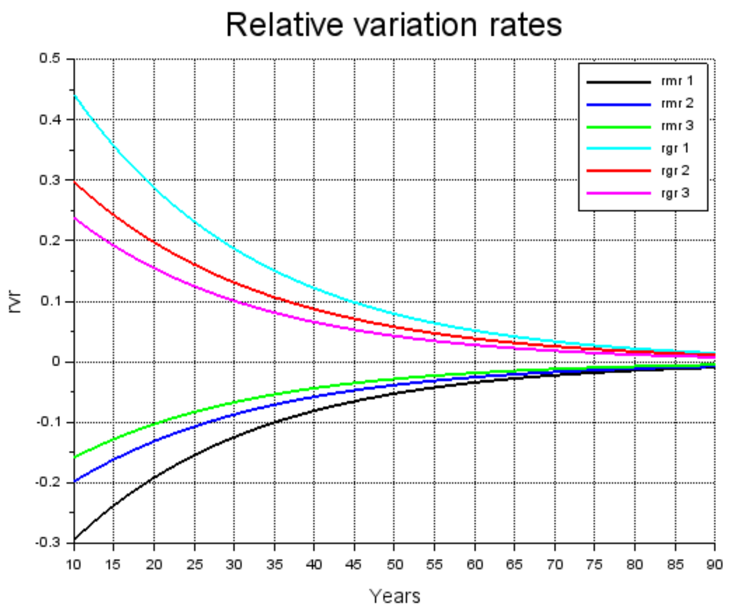
"Over or underyieldig"

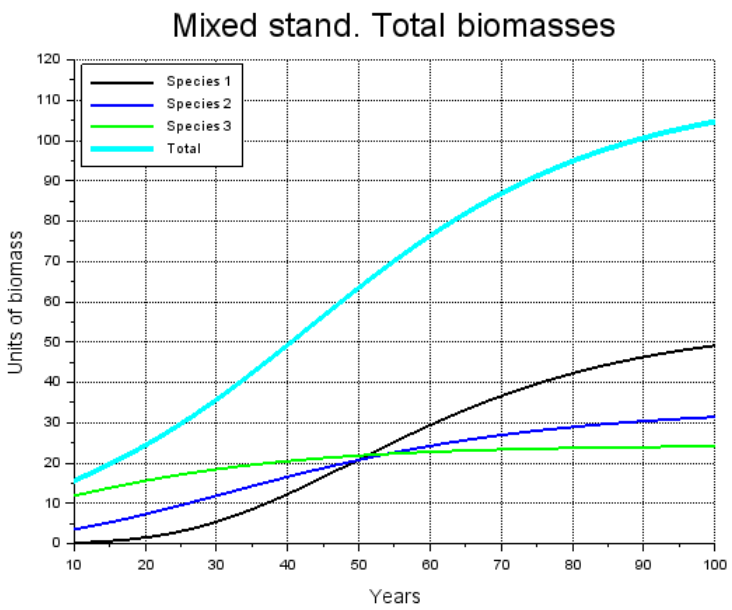
WARNING: Transposing row vector X to get compatible dimensions

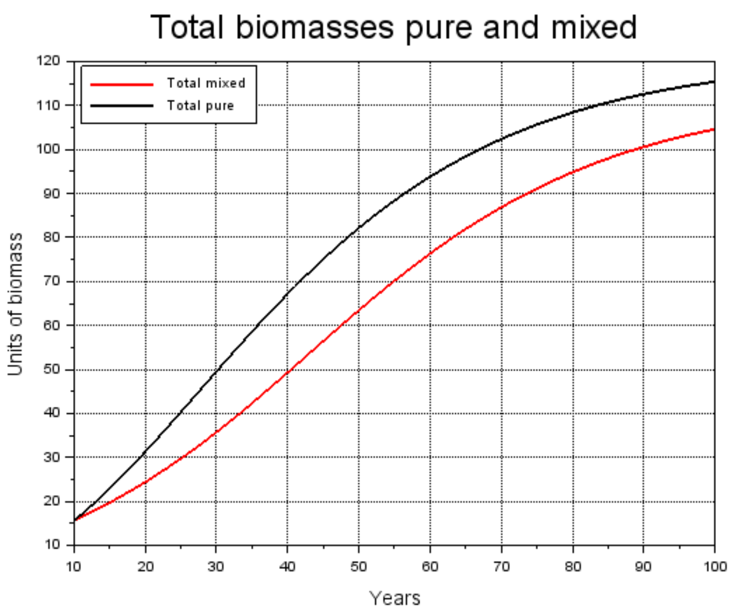
"Over- or underyielding"

ans =

0.908 ← **Underyielding occurs as evinced in the third graphic**




Mixed stand. Total biomasses
Species 1
Species 2
Species 3
Total
Units of biomass
Years
0
10
20
30
40
50
60
70
80
90
100
110
120
10
20
30
40
50
60
70
80
90
100


Total biomasses pure and mixed
Total mixed
Total pure
Units of biomass
Years
10
20
30
40
50
60
70
80
90
100
110
120
10
20
30
40
50
60
70
80
90
100

6. Conclusive Remarks

Contrary to what some authors admit, adding a third species to a mixture of two species does not necessarily cause an increasing of overyielding (Pretzsch, 2018:134). This matter is not so simple. Thus, the issue I approached in this text was a problem requiring clarification.

This text illustrates two benefits due to the existence of a scientific theory (Bunge, 1998:436):

- To explain facts by means of systems of hypotheses entailing the propositions that express the facts concerned;
- To increase knowledge by deriving new propositions (e.g., predictions) from the premises in conjunction with relevant information.

This text illustrates the fecundity of my theory for forests (Barreto, 2011), that is a particular case of my mathematical, and unified theory for ecology (Barreto, 2017).

I admit that the general framework, and the two mechanisms I presented are a valuable contribution to the problem I approached.

Two embracing books dedicated to mixed forests are Pretzsch, Forrester, Bauhus (2017), and Bravo-Oviedo, Pretzsch, del Río (2018).

7. References

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Barreto, L. S., 2020. *Overyielding in Mixed Forests Revisited. A Theoretical and Simulative Inquiry.* http://hdl.handle.net/10400.5/20148

Bravo-Oviedo, A., H. Pretzsch, M. del Río, Editors, 2018. *Dynamics, Silviculture and Management of Mixed Forests,* Springer International Publishing AG.

Bunge, M., 1998. *Philosophy of Science. From Problem to Theory.* Volume 1. Transaction Publishing, New Brunswick, U.S.A.

Pretzsch, H., 2018. Growth and Structure in Mixed-Species Stands Compared with Monocultures: Review and Perspectives. In Andrés Bravo-Oviedo, Hans Pretzsch, Miren del Río, Editors, *Dynamics, Silviculture and Management of Mixed Forests,* Springer International Publishing AG, pages 131-183.

Pretzsch, H., D. I. Forrester, J. Bauhus, Editors, 2017. *Mixed-Species Forests Ecology and Management.* Springer-Verlag GmbH Germany.

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